The Teaching of Arithmetic through Four Hundred Years

¶ An introduction for to lerne to recken with the pen, or with the counters accordynge to the trewe cast of Algorysme, in hole nombers or in broken/ newly corrected. And certayne notable and goodlye rules of false posytions therevnto added, not before sene in oure Englyshe tonge, by the whiche all maner of difficyle questyons may easily be dissolued and assoylyd. Anno dñi. 1539.

Thom. Tanner

TITLE-PAGE OF "AN INTRODUCTION FOR TO LERNE TO RECKEN," 1539 EDITION

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The Teaching of Arithmetic through Four Hundred Years

(1535–1935)

BY

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Dr Ballard and his publishers, the University of London Press, have most kindly allowed me to reproduce a page from *The Child's First Number Book*.

My grateful thanks are due also to the Secretary of the Bodleian Library for his assistance in finding an illustration from *An Introduction for to Lerne to Recken with the Pen*.

The occasional mention of *Arithmetical Books*, by A. De Morgan, as the source of information of a passage does not cover the use which has been made of this valuable book of reference.

F. A. Y.

CONTENTS

CHAPTER I

THE REASON FOR STUDYING ARITHMETIC

> Alle thynges that bene fro the first begynnyng of thynges have procedede, and come forthe, and by resoun of nombre ben formede; And in wise as they bene, so owethe they to be knowene; wherefor in vniuersalle knowlechyng of thynges the Art of nombrynge is best, and most operatyfe.

THIS truth, that number is the basis of all things, was the reason always given in the Middle Ages for the study of number. It stood for eight hundred years in Boetius as the recommendation of a theoretical arithmetic which was largely a discussion of factors, ratio, and proportion, and was repeated early in the thirteenth century by Sacrobosco as a preface to his account of the Hindu-Arabic rules of reckoning. The quotation is the rendering of this preface in the one English translation[1] of Sacrobosco's work.

This work, *De Arte Numerandi*, was the most widely read of the algorisms, or treatises on Hindu-Arabic arithmetic. It gave the few known rules very concisely, without making any suggestion as to the application of them. It was not necessary for Sacrobosco to suggest their use. He was writing for the schools in which arithmetic was known to be a study of the properties of numbers, and reckoning, its practical side, of use in computing the calendar. Merchants reckoned

[1] Ashmolean MS. 396. For full text see *The Earliest Arithmetics in English*, by R. Steele (Oxford University Press). An analysis of the manuscript is in *The Story of Reckoning in the Middle Ages*, by the present writer (Harrap).

by the new methods also; they had brought the Hindu-Arabic rules home with their wares, and by this time were devising a number of applications of the Golden Rule to the problems of trade. Their knowledge was reaching a wide public, but the treatises on arithmetic were as yet only for the privileged few.

In the years of the Renaissance, when many things came under review, arithmetic was rewritten for the ordinary man. It fell to the writers of the early books to choose from the medieval material the topics to include, and to point the direction along which future progress would be made. To Robert Recorde is generally given the credit of determining the fashion of modern arithmetic, because he was the first to write an English book of any popularity. His *Ground of Arts* contained the fundamental rules given by Sacrobosco and the derived rules used in the affairs of the people, making the whole an applied arithmetic. Succeeding writers have occasionally kept a clearer distinction between pure and applied arithmetic, but almost without exception have followed Recorde in emphasizing the applied side of it. Gradually, through four hundred years, owing to modifications of the old rules and the introduction of new topics, these books have been developing into the text-book of to-day.

If we wish to know what guided these earlier teachers and made arithmetic what it is we cannot do better than study the prefaces of their books, where they have generally set out the special considerations which weighed with them when choosing their matter and method. Most of all should we read those of the sixteenth century, that we may understand the conception of arithmetic current when the first books in English were written.

The earliest known book in English on arithmetic, *An Introduction for to Lerne to Recken with the Pen,* recommends arithmetic thus:

> That art and feate (dear Reader) whom utilitie and necessity both do commende needeth greatly of no other commendation. Howe profitable and necessary this feat of Algorism is to all maner of persons, which have reckenyngs or accountes, other to make, or else to receive, needyth no declaration. Neither is this arte only necessary to those, but also in maner to all manner of sciences and artificies. For what craft is that but it some tyme doth occupy not only one part of this feate, but all the partes.

The *Ground of Arts,* 1543, devotes its first chapter to "the declaration of the profit of Arithmetick." In it it says that the use in astronomy, geometry, music, and physics is apparent, in law a judge deals with the distribution of food and debts, in the army there must be provision of victuals, artillery, armours, and wages, and so, mentioning all the trades and professions in turn—merchants, bailiffs, auditors—it shows the special need of each of a knowledge of arithmetic: "With the help of it you may attain to all things."

Thomas Hylles fifty years later gives an even longer list of occupations benefited by arithmetic:

> No State, no age, no man, nor child, but here may wisdome win
> For numbers teach the parts of speech, wher children first begin.
> And number bears so great a sway even from the most to least
> That who in numbring hath no skill, is numbred for a beast:
> For what more beastly can be thoght? nay what more blockish than
> Then man to want the onely art, which proper is to man,
> For many creatures farre excell mankind in many things,
> But never none could number yet, save man in whom it springs.

If numbring then be (almost) al, betweene a man and beast,
Come learne o men to number then, which arte is here perofest
If martial man thou minde to be, or office do expect.
In court or country where thou dwelst, or if thou do elect,
In Phisicke and Philosophie, or law to spend thy dayes,
Assure thyself without this arte, thou never canst have praise.
I overpasse Astronomie, and Geometrie also,
Cosmographie, Geographie, and many others mo,
And musick with her dulcet tunes, all which without this arte,
Thou never canst attayne unto, nor scarce to any part,
Ne canst thou be an auditor, or make a true survey,
Nor make a common reckoning, if numbers be away.
But if thou willte a merchant be, then make this booke thy Muse,
Wher thou shalt find rules fit for thee, as thou canst wish or chuse.
And having onely handie craft, yet herein mayst thou finde,
Such things as may oft serve thy turne, and much enrich thy minde.
Nay if thou but a shepherd be, it wil thee sore accumber
To do thy duty as thou shouldst without the help of number.
To number all the benefits, that number brings to man,
Would be too long here to rehearse, and more than well I can,
Wherefore to speake one woorde for all, and let the rest alone,
Without this art man is no man, but like a block or stone.

By this time, 1600, sheer usefulness had won the day, and the older theoretical arithmetic of Boetius, heavily handicapped from the time algorism was introduced, was left almost unread.

Books keeping the aim of usefulness always in view differed from the manuscripts from which they sprang. Their readers wanted to apply the rules to ordinary commodities measured in terms of money, weights, lengths, etc. So it came about that concrete numbers dominate their pages.

Merchants, we know, had early learned the value of the new written reckoning; and it may be that the contemporary books on the subject were found to be more in demand by men in trade than in other occupations. It is difficult to explain otherwise the commercial bias of the books, which became more pronounced as time went on. John Mellis, a schoolmaster, inserted commercial chapters in Recorde's *Ground of Arts*; and John Kersey, another schoolmaster, edited and emphasized the commercial chapters in Wingate's more balanced book, saying that he did it "for the ease and benefit of those learners who desire only so much skill in Arithmetic as is useful in accounts, trade and such like ordinary employments."

This desire to be of help to men in trade went too far, arithmetic becoming so commercialized that in the seventeenth century it was being neglected except by those whose way in life demanded it. There were many references in the current literature to this lack of teaching of the leisured classes. One man, for instance, writes:

> I never learned but very little Arithmeticke, for I never did learne any for wayte nor for measure wh. ought to be taught Rule by Rule with the other which is money. I never learned but five rules; it is true that I had begunne the 6th which is called ye Rule of 3, but I was never perfect in it.[1]

In *The Young Gentleman's Course in Mathematics* (1714), Edmund Wells, knowing arithmetic only as a useful art yet thinking all should know it, wrote in his foreword:

> Gentlemen should not be so brisk and airy as to think that the knowing how to cast accompts is requisite only for such underlings as shopkeepers or tradesmen. . . .

[1] *Verney Memoirs*, 1657 (Longmans).

> No gentleman ought to think Arithmetic below him who does not think an Estate below him.

These are only two of many who held it a matter for regret that arithmetic had come to be considered an art for trade purposes only; and Edmund Wells, whose book bears the words 'young gentleman' on every other page, and was likely to be laughed at on that account, did arithmetic a service in making it again more general. Nevertheless, until the eighteenth century, no matter for whom it was written, it was the usefulness of it, and only the usefulness of it, that was urged as the reason for the study of arithmetic.

A somewhat wider view was taken by Alexander Malcolm in 1730. On the title-page of his arithmetic he states his double aim: "to make a complete system of theory for the purposes of men of science, and of practice for men of business." In the preface he says:

> Though there be many truths discovered in the Theory of Arithmetic of which there has been no use or application yet found, there is no reason why these things should be neglected or kept out of the system; for they are still a part of the Science which we ought to enlarge more and more as far as we can: one age may find the use of the Theory which a former has invented. . . .
>
> The mind of man is made for knowledge and contemplation, and the pleasure arising from the perception of Beauty and order in other things is allowed to be worthy of rational natures: the contemplation of the surprising connections, the beautiful order and harmony of relations and dependencies found among numbers, is not less reasonable. . . .
>
> Others ask no more than plain rules for the practice as far as they have use for it.

With this body of opinion expressed through many years, nobody questioned the place of arithmetic with reading and writing in the curriculum of schools. The

form arithmetical teaching was to take was established; and it remained for teachers only to continue to watch the times, and strengthen whatever section of it the interests of the day seemed to warrant. Thus, when Government raised money from the people, or companies were promoted, stocks and shares were introduced; in modern times, when results are presented scientifically, we keep pace with statistics and graphs; when patriotism calls we teach arithmetic of citizenship. Though some see the æsthetic side of numbers, Boetian arithmetic has shrunk to one chapter on primes and factors. Despite what Malcolm wrote and many think, arithmetic for the present holds its own on the grounds of what is presumed to be its immediate usefulness.

The increased number of national schools made it necessary after 1800 to give more consideration to arithmetic as a school subject. The chief concern since then has been how it should be presented. One early criticism of the prevalent way of teaching it came from Thomas Clark in 1812. He said that there was not a work of any repute whatever in the English language in which the four fundamental rules were clearly and comprehensively laid down and given with examples enough to remove the difficulties which these rules must present to beginners, or one in which the rationale of arithmetical operations seemed of sufficient importance to the instructor to induce him to incorporate it in his work. De Morgan also was anxious to improve the teaching of elementary arithmetic, and in 1830 he brought out a book to point the way. He put on his title-page a quotation from Condillac:

> Ce n'est point par la routine qu'on s'instruit, c'est par sa propre réflexion; et il est essentiel de contracter

l'habitude de se rendre raison de ce qu'on fait; cette habitude s'acquiert plus facilement qu'on ne pense; et, une fois acquise, elle ne se perd plus.

De Morgan was able to say, however, when bringing out the fifth edition of this book in 1854:

> At the time when this work was first published, the importance of establishing arithmetic in the young mind upon reason and demonstration was not admitted by many. The case is now altered: schools exist in which rational arithmetic is taught, and mere rules are made to do no more than their proper duty. There is no necessity to advocate a change which is actually in progress, as the works which are published every day sufficiently show.

Another who saw need for reform was Edward Sang. In the preface to *Elementary Arithmetic* (1856) he wrote:

> It seems as if this subject were never regarded as having in it anything intellectual. Arithmetic is considered as a kind of *legerdemain*, a talsamic[1] contrivance, by means of which results are to be obtained in some occult manner, into the nature of which the student is forbidden to enquire, . . . there is hardly any branch of human knowledge which affords more scope for intellectual effort, or presents a more invigorating field for mental exercise than the science of number. It has therefore been the author's aim to prepare a text-book which should call mind, not memory, into exercise, and from which all mere dogmatism should be scrupulously excluded.

These few extracts show the trend of opinion in the nineteenth century, the century of reform.

A succession of writers have thus told us, in the earlier years, why they wrote of arithmetic, and, later,

[1] I cannot trace this word in dictionaries. Misprinting for "talismanic" suggests itself, but it may be an allusion to something now forgotten.

how they were hoping to improve the teaching of it. Their books are described in the following chapters.

It is difficult to assess the worth of the older books. We know them to be written for learners of a different age and outlook from those of school pupils of to-day, and, moreover, for learners who often read alone or with a tutor. A teacher of to-day would probably put Wingate first among those text-books written before the nineteenth century, but it was not the favourite; nor did it last nearly as long as Cocker. Speaking broadly, there was a series of good books from 1540 to 1670; then in 1677 Cocker led in a series of poor books; and later, near the turn of the eighteenth to the nineteenth century, an improvement took place, becoming more pronounced as the century advanced. As a rule the writers left it to others to declare the success of their work. One seventeenth-century writer, however, forestalled his critics. It was characteristic of John Hawkins, the editor of Cocker's two arithmetics, to overburden the books with prefaces, and one of these, in *Decimal Arithmetick*, praising the one-century-old decimals and the author of the book simultaneously, makes this extraordinary claim:

TO ... THE FAMOUS ARITHMETICIAN ... EDW. COCKER

With Admiration struck, I here would pause,
Not daring trust my Muse in your Applause;
Whose fame already has so loud been sung
By the Divinest of the Sacred Throng:
Did not your rich and matchless Art inspire
My drowsy Soul with a Poetick Fire.
For who in Silence can remain, that views
A Subject worthy, such as can infuse
A moving Rapture of the first degree
Into a Breast, before from Phœbus free?
So great a masterpiece as this, Mankind
In all their tedious Search could never find.
Arithmetick's here to Perfection brought;
Here's to be found what never yet was taught,

The curious work so to the Life is drawn,
That all besides are like the Morning's Dawn,
Compar'd to Day's clear Face when Sol sits high
In his Meridian Throne; in vain some try
To reach your Art's Perfections, but the more
Their Genius flags when to your Heights they'd soar;
And at the best their Labours do appear
Foils to make your Diamonds shine more clear.
This book of yours bears Record of your Fame,
And to all Ages will transfer your Name.
For why, your boundless Wit, and curious Pen
Do still write you the Miracle of Men.

CHAPTER II

TONSTALL

ENGLAND is fortunate in having the earliest Western manuscript on Hindu-Arabic reckoning. It is the Cambridge Manuscript I, i, 6, 5, and belongs to the very early years of the twelfth century. It is even possible that this manuscript introduced the new reckoning into the country. It contains a careful account of numeration and very brief explanations of simple cases of the fundamental rules. There are only eight leaves—that is, sixteen pages—of writing, the work breaking off in the middle of a sentence at the foot of a page. Evidently part is lost.

In the next century the better known *De Arte Numerandi* of Sacrobosco explained numeration, addition, subtraction, mediation, duplication, multiplication, division, arithmetical progression, and extraction of square and cubic roots. Only very simple cases of each were given, and the directions were clear and orderly. Another popular text was *Carmen de Algorismo*, by Alexandre de Ville-Dieu, giving the rules in verse. These treatises were the basis of the work done in the various schools, as apart from the market-place, where also arithmetic was practised. Here and there a manuscript indicates the progress made, but how much the subject developed before the popular arithmetics were written we could scarcely have known were it not for the *De Arte Supputandi* of Tonstall, published in 1522.

Cuthbert Tonstall became Bishop of London, and

the book was completed only a few days before his consecration. He says in the foreword that he had read everything he could on the subject in every language he knew, and then had spent much time putting it together. The result is a classic work and a book to which we may go with confidence to learn the state of the fullest knowledge of Hindu-Arabic arithmetic at the beginning of the sixteenth century. De Morgan thinks that the book was very little known to subsequent writers. Be that as it may, the sources were open to all students. Only occasional reference is made to it, and only very rarely does a book give the impression of drawing from Tonstall, but nearly the same range of rules and some of the same problems are found in later books.

In Tonstall the parts of arithmetic were reduced to seven: numeration, addition, subduction or subtraction, multiplication, partition, progression, and extraction of roots.

At the beginning of the book Tonstall explained the Hindu-Arabic method of writing numbers, which was still considered to be the new notation, though four centuries had passed since its introduction into England.[1] The usage during these centuries, and until the popular arithmetics were read, was to write a number in the Roman notation, and to reckon with counters. The new notation could be used for both the reckoning and the permanent record of the result. The explanation of the notation has varied little in the centuries, and consists of three essential parts, namely, the description of the nine symbols and their values, the explanation of *local value*, and the explanation of the value and use of the cipher.

For easy reading of a long number Tonstall put

[1] See *The Story of Reckoning in the Middle Ages.*

points over the fourth, seventh, tenth, etc., figures, reading from the units figure leftward—*e.g.*,

$$\dot{1}23\dot{4}56\dot{7}89\dot{0}123$$

In later years the superior points were removed, and the grouping of the figures into threes was shown more clearly by means of commas placed to the right of the fourth, seventh, tenth, etc., figures—thus,

1,234,567,890,123

Addition and subsequent rules begin with a definition. Numbers to be added are arranged as now in rows, with their digits of like place value in columns. Good explanations are given of carrying figures and ciphers, with numerous examples to cover all possible cases. There is a triangular table of added digits, of which the first two lines are these:

9	9 9 18	9 8 17	9 7 16	9 6 15	9 5 14	9 4 13	9 3 12	9 2 11	9 1 10
8		8 8 16	8 7 15	8 6 14	8 5 13	8 4 12	8 3 11	8 2 10	8 1 9

One example is given of addition of concrete numbers:

AUREI	DENARII	SESTERTII
904	15	6
243	16	8
175	19	9
1324	5	3

The table of money is: 1 aureus = 25 denarii, 1 denarius = 4 sestertii.

In subtraction the rule given for applying when the

upper digit is the smaller is: add ten to the top-line figure and adjust on the figure one place to the left below.

There is a table for subtraction, the first two lines of which are these:

	19	18	17	16	15	14	13	12	11
9	10	9	9	9	9	9	9	9	9
	9	9	8	7	6	5	4	3	2
			17	16	15	14	13	12	11
		8	8	8	8	8	8	8	8
			9	8	7	6	5	4	3

This table is intended to guide the reader through the difficulty often encountered in a subtraction. It shows the figure which should be set down in the remainder for each case in which the figure of the subtrahend is greater than the figure of the minuend. There is one case shown in which the subtrahend ends in a cipher, and the figure of the minuend appears in the remainder. For many years the cipher, 0, and often the unit, 1, were not considered to be numbers, and it is only by linking the cipher with a digit and so making a number that Tonstall could show how to deal with it in subtraction. One case was sufficient, and for economy in space it was put in the line of subtrahend 9.

The example of subtraction in concrete numbers is:

Aurei	Denarii	Sestertii
1123	12	2
456	17	3
666	19	3

Multiplication begins at the right-hand end, and the figures of the multiplier are crossed out as used. The

table for multiplication is the square Pythagorean table, in which the first line and column are the natural numbers up to ten, the second line and column are the doubles of the numbers, and so forth.

For division or partition this table is reversed. The dividend is at the intersection of a line and a column, and the divisor and quotient are the numbers at the right-hand end of the line and at the bottom of the column respectively.

100	90	80	70	60	50	40	30	20	10
90	81	72	63	54	45	36	27	18	9
80	72	64	56	48	40	32	24	16	8
70	63	56	49	42	35	28	21	14	7
60	54	48	42	36	30	24	18	12	6
50	45	40	35	30	25	20	15	10	5
40	36	32	28	24	20	16	12	8	4
30	27	24	21	18	15	12	9	6	3
20	18	16	14	12	10	8	6	4	2
10	9	8	7	6	5	4	3	2	1

A division looks strangely confused, but if the reader will divide 564321 by 1200 he will discover a

$$\begin{array}{r|l|l} 321 & \cancel{1}\cancel{8} & \\ 1200 & \cancel{5}\cancel{6}\cancel{4}3 & 21 \\ \hline & 470 & \\ \hline & \cancel{1}\cancel{2}\cancel{2}2 & \\ & \cancel{1}\cancel{1} & \end{array}$$

quotient of 470 and a remainder of 321. In the above example the two figures 21 are cut off by a line because the divisor is an exact multiple of a hundred. The resulting divisor, 12, is placed below the line and moved at each step under the figures of the dividend it is then dividing, though it may be spread over two horizontal lines. The working is 12 × 4 = 48. Subtract 4 from 5 and there remains 1. Cross out 5 and put up 1. Subtract 8 from 16 and there remains 8.

Cross out 1 and 6 and put up 8, etc. The remainder 321 is set over the divisor at the extreme left-hand side of the problem.

Subtraction from the left as here was not a universal practice in division, but has always to be looked for as a possible procedure. It is not adhered to in the following example.

The inconvenience of the massed figures is not felt as the work proceeds, because the eye does not rest on the figures struck out.

```
          129 11
         |106055239
     1397|1369703945 7
     1579|76859463820
          ------------
          48676037
          ------------
          15799999999
           157777777
            1555555
             11111
```

Tonstall used short division where the divisor was a digit, but many years elapsed before the method appeared in the English books.

```
 1|
 -|
 2|1141031
   -------
   570515
```

As in the cases above the remainder, 1, is placed at the left-hand side over the divisor.

The chapter on progressions is short. The only series given begin with unity and have differences of 1, 2, and 3. The sum is found, as in Sacrobosco, by rules based on the formulæ $\frac{n}{2} \times (a + l)$ and $n \cdot \frac{a + l}{2}$, according as n is even or odd.

The extractions of roots, square or cubic, are arranged as in division, the successive divisors below the line and the root between the lines gradually appearing as the given number above disappears. The square root of 57836029 is here shown to be 7605 with a remainder 4.

```
 8 7   4
57836029   Reliquum
--------
 7 6 0 5   Radix
--------
 145220
  113
```

The preparation of a number for the extraction of its square root consists of the grouping of the figures into sets of two, counting from the units figure. Tonstall marks the groups by placing a point over the first figure in each group, thus $5\dot{7}8\dot{3}6\dot{0}2\dot{9}$. A modern method is to use commas, thus, 57,83,60,29.

This completes Book I, the ground covered being the same as in Sacrobosco's treatise. The further books are additional to the teaching of Sacrobosco.

Book II begins with fractions. There are good explanations of finding equivalent fractions by reducing to lower terms or bringing up to a higher denominator; also of reducing an improper fraction to a mixed number. These preliminaries over, addition and subtraction present no difficulty. For the multiplication of two fractions the rule is to multiply the two numerators for the new numerator and the two denominators to get the new denominator. If necessary the result may be put into a mixed number, but there is no cancelling during the process. There is also no inversion of the divisor in division, the fractions standing as they are given and the quotient being obtained by cross-multiplication. This needs care as the divisor is often put first in such phrases

as "*si per* $\frac{1}{2}$ *secces* $\frac{1}{3}$: *prodeunt* $\frac{2}{3}$" and "*sic per* $\frac{3}{8}$ *divise* $\frac{2}{3}$ *faciunt* $\frac{16}{9}$." There are several examples worked through illustrating different cases that may arise; for instance, the divisor may be an integer, or a fraction larger than the dividend.

In the miscellaneous work that follows this chapter there is a figure of a square of side 6 units in length, cut into four parts, two squares and two rectangles, by two lines, each parallel to one pair of sides, and dividing the other pair of sides into two parts of 2 units and 4 units. A figure corresponding to this to-day would contain in addition lines cutting the sides of the square each into six parts 1 unit in length, and the surface into 36 squares of unit area. Tonstall marks the approximate position of the smaller squares with series of 1's, and in his figure we can see by counting that the composition of the square is a square of 4, or 2×2, units of area, another square of 16, or 4×4, units of area, and two rectangles each of 8, or 2×4, units of area.

1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1
1	1	1	1	1	1

The rule for extracting a square root is given in Book I. On Tonstall's own assurance that he had put into this work everything on arithmetic he could find, we must believe that it was not the custom then, even

in oral teaching, to analyse a square number and explain the steps of evolution with continual reference to this square.

The book ends with numerous examples, first grouped for solving by addition, subtraction, multiplication, and division, and then miscellaneous questions to be solved by 'many and divers' rules.

Book III is on proportion. A distinction is drawn between *continual* and *separate* proportion, the whole work leading up to the rule for finding the fourth term by dividing the product of the second and third terms by the first. When this rule comes to be applied to concrete numbers in problems the proportion is always stated intelligibly, that is, between quantities of like nature—*e.g.*, a number of books is compared with another number of books, a number of angels with another number of angels. In the Rule of Three as it was taught afterwards the terms were kept in the order in which they appeared in the question, a practice which invariably resulted in a meaningless statement. Comparisons such as '3 horses to £20' were the rule. It happens that a few rules have found their way into the arithmetic books—the Rule of Three, with its misplaced second and third terms, practice, the duodecimal rule of area, and bankers' discount—which do not bear on the face of them the stamp of the mathematician. The last three mentioned are admittedly the merchant's, the decorator's, and the banker's contribution to arithmetic, but we cannot dismiss this Rule of Three so easily by saying that it is merchant's practice. A study of the English books points rather to a deterioration in the teaching of it, due in the first place to omission to make a statement of the proportion in words and in later years to lack of clear thinking when stating the proportion.

Before he leaves this subject Tonstall takes the rule of proportion with fractional numbers.

The next rule in applied arithmetic concerns the sharing out of profits. The principle is explained with two simple cases in which three merchants put differing sums of money in a business, the first example bringing out their shares in whole numbers, the second in fractional numbers. The rule is extended to partnership taking time into consideration. Many cases of sharing are explained, some quite intricate. Question 16 is the familiar question of the twins. The shares vary in different books, but here it is given that if a son is born he takes two-thirds and the mother one-third of an inheritance, whereas if a daughter is born the mother takes two-thirds and the girl one-third of the money. As both a boy and a girl are born, the testator's intention is interpreted as requiring that son, mother, and daughter take shares in the proportion 4 : 2 : 1.

As the questions advance they become miscellaneous, and depend on any of the rules hitherto taught. Question 27 is of the full cistern with three pipes which can empty it in one, two, and three hours respectively, and the time to empty it, employing all these simultaneously, is sought. Another asks the meeting-place between London and York of travellers of differing speeds. There are also problems on mensuration, and one on the distance round a circle, in which the ratio of circumference to diameter is given as 22 to 7.

A considerable portion of Book IV discusses series and proportion. It is an analysis of the properties of numbers in relation to one another. The earlier pages deal with arithmetic, geometric, and harmonic series, giving first a definition and explanation of each and

then the method of finding the means. An introductory and very elementary chapter on progressions appears in Book I. In this later book the series may begin with any number and ascend, that is, the difference in arithmetic series, or the ratio in geometric series, is always a positive integer.

The book ends with the Rules of False Position. The rules did not change to any great extent up to the time in the nineteenth century when they were discarded and the problems they used to answer were solved by simple algebraical equations.

An example, showing the Rule of Double False Position, taken from Cocker's Arithmetic, written about one hundred and fifty years after Tonstall's, is worked in Chapter V of this book.

At one point in the Rule of False Position the worker has to determine whether to proceed by addition or subtraction. The directions for choosing were generally put in verse for memorizing; and among the few we find in later books Recorde's verse is most nearly like Tonstall's. Four lines of Recorde's 'obscure riddle,' quoted in Chapter III of this book, are a paraphrase of Tonstall's verse:

A plure deme plusculum,
Minus minori subtrahe,
Pluri minus coniungito,
Atque ad minus plus adjice.

CHAPTER III

RECORDE

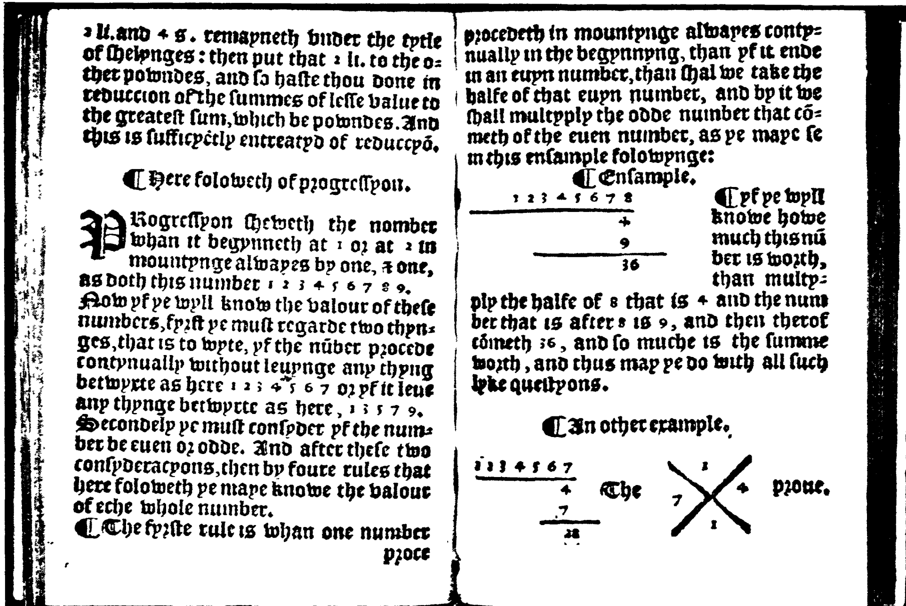
"AND whereas, before, our forefathers had no other books but the score and the tally, thou hast caused printing to be used."[1] This might aptly have been written of Robert Recorde, whose *Ground of Arts* brought written arithmetic measurably nearer to those who had only score and tally, fingers and counters. Before his time there were, as we have seen, the Latin manuscripts leading up to the *De Arte Numerandi* of Sacrobosco, which by 1523 had been printed at least five times, a poem, *Carmen de Algorismo*, and Tonstall's classic book. There were in addition to these a very few manuscript treatises in English, of which the best known are *The Crafte of Nombrynge* and the translation of *De Arte Numerandi*.[2] A fourteenth-century treatise on numeration,[3] an unfinished fragment on the arithmetical rules,[4] and other small writings on arithmetical topics in English are also known. There was one book in English. The first appearance of this book, *An Introduction for to Lerne to Recken with the Pen*, is said to have been at St Albans

[1] Jack Cade (in his indictment of Lord Say): "Thou hast most traitorously corrupted the youth of the realm in erecting a grammar school: and whereas, before, our forefathers had no other books but the score and the tally, thou hast caused printing to be used; and, contrary to the King, his crown and dignity, thou hast built a paper mill. It will be proved to thy face that thou hast men about thee that usually talk of a noun and a verb, and such abominable words as no Christian ear can endure to hear."—*Second Part of King Henry VI*, Act IV, Scene 7.

[2] See *The Earliest Arithmetics in English*.

[3] See *Rara Mathematica*, by J. O. Halliwell.

[4] See *The Earliest Arithmetics in English*.



2 li. and 4 s. remayneth vnder the tytle of shelynges: then put that 2 li. to the other powndes, and so haste thou done in reduccion of the summes of lesse value to the greatest sum, which be powndes. And this is suffycyently entreatyd of reduccyō.

¶ Here foloweth of progressyon.

Progressyon sheweth the nomber whan it begynneth at 1 or at 2 in mountynge alwayes by one, & one, as doth this number 1 2 3 4 5 6 7 8 9. Now yf ye wyll know the valour of these numbers, fyrst ye must regarde two thynges, that is to wyte, yf the nūber procede contynually without leuynge any thyng betwyxte as here 1 2 3 4 5 6 7 or yf it leue any thynge betwyxte as here, 1 3 5 7 9. Secondely ye must consyder yf the number be euen or odde. And after these two consyderacyons, then by foure rules that here foloweth ye maye knowe the valour of eche whole number.

¶ The fyrste rule is whan one number proce-

procedeth in mountynge alwayes contynually in the begynnyng, than yf it ende in an euyn number, than shal we take the halfe of that euyn number, and by it we shall multyply the odde number that cōmeth of the euen number, as ye maye se in this ensample folowynge:

¶ Ensample.

1 2 3 4 5 6 7 8
4
9
36

¶ Yf ye wyll knowe howe much this nūber is worth, than multyply the halfe of 8 that is 4 and the number that is after 8 is 9, and then therof cōmeth 36, and so muche is the summe worth, and thus may ye do with all such lyke questyons.

¶ An other example.

1 2 3 4 5 6 7
4
7
28

The proue.

1 7 4 1

PAGES FROM "AN INTRODUCTION FOR TO LERNE TO RECKEN," 1539

The pages show the beginning of the discussion of the rule of progression.

By courtesy of the Bodleian Library, Oxford

in 1537. An early edition, dated 1539, is preserved in the Bodleian Library. It gives clear explanations of the seven parts of arithmetic and a few derived rules. The book was enlarged by 1546, and bears the long title, in part descriptive of the contents, *An Introduction for to lerne to recken with the pen, or with the counters according to the trewe cast of Algorisme, in hole numbers or in broken and certayne notable and goodly rules of false position thereunto added, not before sene in our Englyshe tonge, by which all maner of difficile questions may safely be dissolved and assoyled.* This edition is in the British Museum. Other editions appeared in 1574, 1582, and 1629, but it was Recorde's *Ground of Arts*, 1543, which took the field and held it, with only one serious rival, Humphrey Baker's *Well Spryng of Sciences,* for over a century. During these years the *Ground of Arts* was frequently brought up-to-date, and Dee, Mellis, Hartwell, and Willsford in turn may take some credit for its long popularity. It was last published in 1699.

The *Ground of Arts* comes as a pleasant surprise to those who know the previous literature. The task Recorde set himself was to select and arrange the material into a sound study of reckoning and to write it up in simple form. He intended his book to be acceptable to "such as shal lacke instructers, for whose sake I have plainly set forth the examples as no booke (that I have seen) hath hitherto: which thing shall be great ease to the rude readers." He showed how the rules were used in the ordinary occupations of the people, and so began the long series of books of applied arithmetic which has continued into the present century as the ideal for primary teaching. He wrote in a dialogue because he thought it the "easiest way of Instruction, when the scholar may

ask every doubt orderly and the master may answer to his question plainly."

A great part of the interest of the book lies in the conversation between Master and Scholar, which reveals the boy's eagerness, the admiration of both for the subject, the idea that it must not be divorced from everyday experience, and the master's vindication of his right to set problems in nonsensical terms.

Scholar. What? this is very easie to doe, methinks I can doe it even since.

. . . .

Scholar. Truly, Sir, these excellent conclusions do wonderfully more and more make me in love with the Art.

Master. It is an Art that the farther you travell, the more you thirst to go on forward. Such a fountain, that the more you draw, the more it springs. And to speak absolutely, in a word (excepting the study of Divinity which is the salvation of our souls) there is no study in the world comparable to this for delight in wonderful and godly exercise: for the skill hereof is well known to have immediately flowed from the wisdom of God into the heart of man, whom he hath created the chief image and instrument of his praise and glory.

. . . .

Master. Yes, but you must prove yourself to do some things without my aid, or else you shall not be able to do any more than you are taught. And that were rather to learn by rote (as they call it) than by reason.

. . . .

Scholar. . . . that is 34952 pounds 10 shillings and 7 pence ob [*i.e.*, a halfpenny].

Master. That is well done; but I think you will buy no horse of the price.

Scholar. No, Sir, if I be wise.

. . . .

Master. As in example, 'If I have bought 30 yards of cloth of two yards breadth, and would have canvas

three yards broad to line it withal, how many yards shall I need?'

Scholar. Why, there is none so broad.

Master. I do not care for that; I do put this example only for your easy understanding: for if I would put the example in other measures, it would be harder to understand. But now to the matter . . .

The book used here is a late edition, 1668, and on the title-page it is described as *Robert Record's Arithmetick, or the Ground of Arts afterwards augmented by Mr John Dee and since enlarged with a third part of Rules of Practice . . . with divers rules incident to the Trade of Merchandise, with Tables of Valuations of coins by John Mellis. And now diligently perused, corrected, illustrated, . . . with an appendix of figurate numbers and the extraction of their roots . . . with tables of Board and Timber measures and new Tables of Interest . . . and Annuities calculated by R. Hartwell.*

There is also a second appendix by Thomas Willsford.

In this edition the two original dialogues are in black letter, and are upon numeration, the first four rules and their application to reductions, progressions, the golden rule, and fellowship, and vulgar fractions used in these rules. There is also a chapter on reckoning with counters.

Numeration, particularly the use of the cipher, is explained fully. A large number is pricked off into threes called *ternaries* or *trinities*. The division of numbers used in finger symbolism, *digits* and *articles*, is mentioned, although it is not used in the book.

After one short illustration of addition the boy sets himself a problem: "There came through Cheapside two droves of cattell: in the first was 848 sheep and

in the second was 186 other beasts." He writes 14 under the first column, and is corrected.

> *Master.* Not so, and here you are twice deceived. First in going about to adde together two summes of sundry things; which you ought not to doe except you seek onely the number of them, and care not for the things: For the summe that shall result to that addition should be a summe neither of sheep nor of other beasts, but a confused sum of both . . . deceived in another point, and that was in writing 14, which came of 6 and 8, under 6 and 8, which is impossible, for how can two figures of two places be written under one figure and one place?

The master thus insists on care in the denominations of numbers, but this generally needed lesson is strained, for the boy had in the first place classed all as "cattell."

To test addition, or prove it, as it was said, casting out nines is recommended, or adding the quantities in two parcels and comparing the total of the two parcels with the first total.

There is addition of money and simple weights and measures in this chapter.

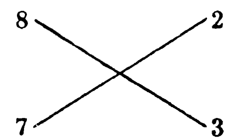
The directions in subtraction are:

> If any figure of the nether summe be greater than the figure of the summe that be over him, . . . then must you put 10 to the over figure . . . and out of the whole summe withdraw the lower figure . . . Whensoever you do put ten to any figure of the over number, you must adde one still to the figure or place that followeth next in the nether line.

Explanation is not asked or given. It is a clear direction for the method of equal additions without the confusing use of the word 'borrow.'

A rule is given for multiplying the larger digits—6 by 6 up to 9 by 9. To multiply 7 by 8, for instance,

place the 7 and 8 at the ends of a diagonal cross and their differences from 10 opposite. This St Andrew's cross is often, as here, associated with multiplication, without bearing its recent meaning 'multiplied by.' Then:



> Last of all, I multiply the two differences saying 2 times 3 is 6; this must I ever set under the differences beneath the line: then must I take one of the differences (which I will, for all is like) from the other digit (not from his own) as the lines of the cross warn me, and that is left must I write under the digits, so 56.

In medieval days, when multiplication was done on the fingers, this, "the dunce's rule," took the form: Number the fingers of each hand, beginning with the thumb, from 6 to 10. To multiply 7 by 8, touch finger 7 of one hand with finger 8 of the other. For the tens digit of the product count and add the fingers up to and including the touching fingers—*i.e.*, $2 + 3 = 5$. For the units digit multiply together the fingers beyond—*i.e.*, $3 \times 2 = 6$. Carrying figures enter into the calculation only in 6 times 6 and 6 times 7.

The multiplication table is triangular:

1	1	2	3	4	5	6	7	8	9
	2	4	6	8	10	12	14	16	18
		3	9	12	15	18	21	24	27
			4	16	20	24	28	32	36
				5	25	30	35	40	45
					6	36	42	48	54
						7	49	56	63
							8	64	72
								9	81

Long multiplication is arranged as it is now, with work beginning from the right. Each digit of the multiplier is crossed out as used.

Division shows even more crossing out. The division of 136,280 by 452 is worked thus:

$$\begin{array}{c}
2 \\
\cancel{1}\cancel{1}\cancel{6}28 \\
\cancel{1}\cancel{3}\cancel{6}2\cancel{8}\cancel{0}(301 \\
\cancel{4}\cancel{5}22\cancel{2} \\
\cancel{4}\cancel{5}\cancel{5} \\
\cancel{4}
\end{array}$$

This differs from Tonstall mainly in the position of the quotient.

The remainder at the first step of the division is 6. The presence of the figure 1 over both the figure 3 and the figure 6 shows that Recorde is beginning the multiplication of 452 by 3 at the left-hand end, subtracting at the same time. Thus, 4 × 3, or 12, is subtracted from 13, 5 × 3, or 15, is taken from 16, and finally 2 × 3, or 6, from 12. At the next step 452 × 1 is 452, which is before him on the paper, and so he is able to proceed without setting down the intermediate figures of the subtraction.

The Master shows the Scholar the alternative, modern 'long' division, "which," he says, "I first learned of, and is practised by, that worthy mathematician, my ancient and especial loving friend, Master Henry Bridges, wherein not any one figure is defaced, or cancelled." A similar remark occurs a little later when Recorde gives more rules in progressions than any predecessor, even Tonstall. Four out of his six rules he says "were invented by a friend of mine and never before this published, and the two first were never to my knowledge written of but by three men." Remarks like these draw a picture of a little group of men keenly interested in this new science—"the farther you travell, the more you thirst

And then muſt I look how often I may find the laſt figure of the diviſor (that is 4) in 13, which I may doe 3 times, therefore do I ſay, 3 times 4 is 12, which I take out of 13, and there remaineth 1. Then do I make at the right hand of my ſummes a crooked line, and write before it my quotient 3, and I cancell 13 and 4, and over the 3 I ſet the 1 that remaineth: and then the figures ſtand thus.

1
~~1~~~~3~~6280 (3
~~4~~52

Then I multiply the ſame quotient into every figure of the diviſor, and withdraw the ſumme that amounteth out of the numbers over them: as firſt I ſay, 3 times 5 make 15, which I take from 16, and there reſteth 1: I cancell therefore 16 and 5, and write over the 6 that 1 that remaineth, thus.

~~1~~1
~~1~~~~3~~~~6~~280 (3
~~4~~~~5~~2

Then doe I ſay likewiſe, 3 times 2 make 6, which I take out of 12, and there reſteth 6; therefore I cancell the 12 and the 2 over, and then I write the 6 that remaineth, thus.

~~1~~~~1~~6
~~1~~~~3~~~~6~~~~2~~80 (3
~~4~~~~5~~~~2~~

Then ſhould I ſet forward the diviſor into the next place toward the right hand thus.

~~1~~~~1~~6
~~1~~~~3~~~~6~~~~2~~80 (3
~~4~~~~5~~~~2~~2
45

Maſter. But you may ſee that over the 4 is no figure, therefore I muſt ſet the diviſor yet forwarder by another place.

And mark, Whenſoever it chanceth ſo that you ſhould ſet forward the diviſor, and that it cannot ſtand here, becauſe there is no number over the

Page from "The Ground of Arts," by Robert Recorde, 1668 Edition

The scholar with the help of the master explains the early steps in the division of 136,280 by 452.

to go on forward"—and working to develop it, and Recorde, one of them, public-spirited enough to pass on the knowledge to others.

In *De Arte Numerandi* mediation and duplication were separate rules. In course of time it was realized that these processes were merely the simplest cases of division and multiplication respectively, and both Tonstall and Recorde make the ordinary rules of multiplication and division cover the multiplication and division by two.

Multiplication by powers of ten is done by adding ciphers, and division by powers of ten by cutting off figures at the right-hand end. 3648 divided by ten is $364\lfloor 8$ in the black letter dialogue, and in the chapter on factorage contributed by John Mellis it is 364|8, with the sub-horizontal line omitted. As is to be expected, these lines were taken over in the notation of decimal fractions.

In these fundamental rules, and throughout the book, the plan is first to define the topic and then to give a plainly worded rule. The illustration is often given simultaneously with the rule, which makes for easier understanding. Proofs or tests of accuracy are given wherever possible.

The greater part of the chapter on reduction is given up to tables of money, weights, and measures, and much of this was added in the seventeenth century. There are tables of the gold coins of Christendom valued in English money for the year 1630 and the prices which foreign coins bring at the Mint according to the Proclamation of May 14, 1612. There were, we learn, in addition to the several gold coins of England the following coins: the Edward Crown (5*s*.), half-crown (2*s*. 6*d*.), shilling, half-shilling, and threepence; Philip and Mary's shilling and half-shilling; the Mary

groat and two-pence; Queen Elizabeth's shilling, 9*d.*, 6*d.*, 4*d.*, 3*d.*, 2*d.*, 1*d.*, three farthings, and half-penny. There is much interesting information on weights for precious metals, wool, cheese, butter, medicines, and the measures for ale, beer, wine, oil, soap, different kinds of fish, grains, salt, lime, and coals. These were defined by law and efforts were being made to secure a uniform practice throughout England. The table of length is substantially that of to-day. In area the shape of the land is taken into account: "1 perch in breadth, 40 in length, do make a Rod of land, which some call a rood, some a yard land, some a Farthendale."

Recorde solves the following six problems of arithmetical progressions:

1. To find the last term, given the first term, the common difference, and the number of terms.
2. To find the number of terms, given the first and last terms and the common difference.
3. To find any middle term, given the first and last terms and the common difference.
4. To find the number of terms, given the first and last terms and the sum of all the terms.
5. To find the common difference, given the first and last terms and the sum of all the terms.
6. To find either first or last term, given the common difference, the number of terms, and the sum of all the terms.

In geometrical progression, whether it is any term or the sum of several that is required, the work always proceeds by multiplication.

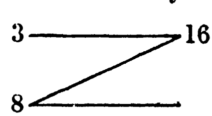
> If I sold unto you a horse having four shoes, and in every shoe 6 nails, with this condition, that you shall pay for the first nail one *ob*, for the second nail two *ob*, for the third nail four *ob*, and so forth, doubling untill the end of all the nails: Now I ask you how much would the price of the horse come unto?

1	1
2	2
4	3
8	4
16	5
32	6
64	7
128	8
256	9
512	10
1024	11
2048	12
4096	13
8192	14
16384	15
32768	16
65536	17
131072	18
262144	19
524288	20
1048576	21
2097152	22
4194304	23
8388608	24

Come to £34952 10*s*. 7½*d*. 16777215 halfpence

The different cases of the Golden Rule are taught as separate rules, a chapter being allotted to each. In all cases the numbers given in the problems are taken apart from their denominations and set as abstract numbers about a symbol. One example from each chapter is appended.

Rule of Three Direct. If you pay for your board for three months sixteen shillings, how much shall you pay for eight months?



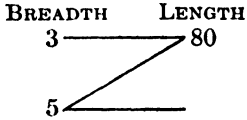
The 3 and 8, being of the same denomination, months, are placed to the same side, and the 16*s*. opposite the 3 with which it is connected.

Rule. Multiply 8 by 16 and divide by 3.

Alternatively he calls 3, 8, and 16 the first, second, and third terms respectively, and multiplies the second and third terms and divides by the first. The terms are put in place and the work proceeds by rote. There is no attempt to read the statement, but with the terms in this, Tonstall's, order the proportion is intelligible.

After this rule the master sets six problems, "of all which questions I omit the work, of purpose that you may whet your wit thereby at convenient leisure." This sentence gives a clue to Recorde's choice of title for his further book containing extraction of roots and algebra, *The Whetstone of Wit.* De Morgan gives a fuller explanation of the title. He writes, "It is rarely remembered that the old name of algebra, the *cossic art* (from *cosa,* thing) gave the first English book in algebra its punning title, *The Whetstone of Wit.*" A history of algebra generally includes the fact that Lucas Pacioli in his book, printed in Venice in 1494, followed the Arabs in referring to the unknown quantity in an equation as the 'thing,' in Italian *cosa,* and hence in England the old name for algebra was the *cossic art.* Recorde's fancy took him from the Italian *cosa,* a thing, to the Latin *cos,* a grindstone or whetstone, and from art, as a study leading to practical skill, to knowledge and wit, and so, combining the two, to his witty title.

The Backer Rule, or Rule of Proportion Backward or Reverse. How many quarters of canvas 5 quarters broad will line 80 quarters of 3 quarters broad?



Reverse the process. Multiply 3 by 80 and divide by 5.

The Double Rule. If the carrying of 100-weight [that is, 112 lb.] 30 miles do cost 12 pence, how

much will the carriage of 500-weight cost carried 100 miles?

100-WEIGHT	PENCE	
1	12	
5		. . . 60 pence.

MILES	PENCE	
30	60	
100		. . . 200 pence.

The Rule of Proportion composed of Five Numbers. This is alternative to the Double Rule. The above example would be worked thus:

100-WEIGHT	MILES	PENCE	100-WEIGHT	MILES
1	30	12	5	100

Multiply the first by the second term, 30, and keep for divisor.

Multiply the third by the fourth and then by the fifth, 6000.

Divide 6000 by 30, the answer coming out in pence.

The Backer Rule Compound. This also is shown in two ways. A merchant received £8 12*s*. for interest of certain money for 5-months term, which he received after the rate of £8 in the 100 for a year. The question is now, how much money was delivered to raise this interest? The proportion is stated thus:

£	MONTHS	£	MONTHS	£	*s.*
100	12	8	5	8	12

This holds three difficulties. It needs close attention to distinguish interest from principal, the double units pounds and shillings are used, and the rule is arbitrary.

It is no wonder that learners before 1570 gave us the rhyme:

> Multiplication is mie vexation,
> And Division is quite as bad,
> The Golden Rule is mie stumbling stule,
> And Practice drives me mad.

Recorde's scholar, however, is made to grasp the rule instantly.

> If it please you to behold me a little, I will quickly end it, for I have but my first, my second and my last number to be multiplied together for my dividend; and my third into my fourth for my divisor.

The Rules of Fellowship, or the division of profits among partners, are applications of the Rule of Three.

The second dialogue opens with thirty-five pages explaining reckoning with counters.[1]

The notation of vulgar fractions is taught, but two of the ways of writing a compound fraction soon became obsolete.

$\frac{3}{4}$ of $\frac{2}{3}$ of $\frac{1}{2}$ may be written as $\frac{3}{4}\frac{2}{3}\frac{1}{2}$ or in a slope thus:

$$\begin{array}{ccc} & & \frac{3}{4} \\ & \frac{2}{3} & \\ \frac{1}{2} & & \end{array}$$

The more confusing former way is used in the book.

The rules of reduction are given as in Tonstall. This enables addition and subtraction of fractions to be disposed of in four pages with the simple direction, for "as you can add 3 pence to 5 pence, in the same denomination, to 8 pence, so fractions must be brought to the same denominator and then you add the numerators."

Fractions to be multiplied are placed in the angles

[1] This is described in *The Story of Reckoning in the Middle Ages.*

of a St Andrew's Cross, with the product of the numerators above and of the denominators below.

$$\begin{array}{ccc} & 15 & \\ \frac{3}{5} & \times & \frac{5}{12} \\ & 60 & \end{array}$$

In division cross-multiplication takes the place of inversion of the divisor.

$$\frac{2}{3} \div \frac{3}{4} \text{ appears as } \quad \begin{array}{ccc} & 8 & \\ \frac{2}{3} & \text{by} & \frac{3}{4} \\ & 9 & \end{array}$$

This example is taken to explain that $\frac{2}{3}$ is to $\frac{3}{4}$ as 8 is to 9, and this definition of division is further illustrated in money. $\frac{2}{3}$ of 1 shilling is 8 pence, $\frac{3}{4}$ of 1 shilling is 9 pence, and therefore $\frac{2}{3}$ is to $\frac{3}{4}$ as 8 is to 9.

The applied rules are then repeated, fractional numbers being introduced. Problems concerning the land and the loaf brought the Backer Rule Simple into the most frequent demand. The acre was defined in two dimensions, but in parcelling out land very often it was convenient to take a rectangle of different shape, in which case the Backer Rule found the length of an acre of given breadth. Also the weight of the farthing loaf, which varied inversely as the price of wheat, was found by the Backer Rule.

There are tables showing dimensions of acres, and weights of loaves when wheat is at 1*s.*, 2*s.*, up to 40*s.* 6*d.* a quarter. Weight is given in pounds, shillings, and pence, as well as in pounds, ounces, and pennyweights. For this latter purpose a shilling is reckoned

as $\frac{1}{20}$ lb. weight, and a penny as $\frac{1}{20}$ oz. There are several corrections and additions in this chapter, because money and commodities changed value in the course of years.

The problem of the twins is in the chapter on fellowship. Recorde's proportions differ from Tonstall's. They are:

The son and mother share as $\frac{1}{2}$ is to $\frac{1}{3}$, or 3 to 2.
The mother and daughter as $\frac{1}{2}$ is to $\frac{1}{3}$, or 3 to 2.

Therefore son, mother, and daughter take shares in the proportion 9, 6, 4.

The Rule of Alligation is the rule of mixtures, and is useful in composing medicines and mixing metals and wines. The Rule of Falsehood was taught until recently, when the surer and shorter methods were shown. Illustrations in these two rules are postponed until Chapters IV and V. Recorde's chapter on falsehood contains two or three points of interest. There is his "obscure riddle," an early example of what later became a vogue, the putting of arithmetical rules into verse. Lines 7 to 10 give the same directions as Tonstall's Latin verse.

Guess at this work as hap doth lead,
By chance to truth you may proceed.
And first work by the question,
Although no truth therein is done,
Such falsehood is so good a ground,
That truth by it will soon be found.
From many bate[1] too many moe;
From too few take too few also;
With too much joyn too few again;
To too few adde too many plaine.
In cross-wise multiply contrary kind,
All truth by falsehood for to find.

[1] Abate = lessen. *Cf.* also rebate = deduction. Here read 'subtract.'

Whenever in working the number guessed proves too great it is marked with the sign +, and a number which is too small is marked with a long dash ——. These signs are used in the sense of excess and defect, not as operative signs.

The last and most difficult problem in falsehood is that of Hieron's crown, which is removed to-day to the science laboratory. This ends the second dialogue. The scholar asks to be taught extraction of roots, and is assured, "I will not break my promise, but intend (God willing) to perform it within this three or four months." Recorde's second book, *The Whetstone of Wit*, fulfilled this promise.

Part III of *The Ground of Arts* is really a separate work. It was written a generation later, towards the close of the sixteenth century. It is a commercial arithmetic by John Mellis, the schoolmaster who in 1588 wrote one of the earliest if not the first English book on book-keeping by double entry. It is bound with Robert Recorde's *Ground of Arts* with a common index. Otherwise there is no attempt to merge the two works. Part III has even a separate title-page. The dialogue has disappeared and with it the black letter. Teaching is now of secondary importance; to get an answer, and that speedily, is the new aim. It is also a reference book to which one may turn for tables of exchange values between different countries. In it nothing is left to the intelligence of the reader. When much has been said of aliquot parts we find twelve separate illustrations showing exactly how to reckon interest on a sum of money for 1, 2, 3, 4, . . . 12 months respectively. Also to aid practice, every number of pence from a penny to elevenpence is given as an aliquot part, or as the sum of two or more aliquot parts, of a shilling, and every number of shillings from

one to nineteen as an aliquot part, or as the sum of two or more aliquot parts, of a pound. The other simple fractions of a pound, $\frac{1}{12}$, $\frac{1}{8}$, $\frac{1}{6}$, $\frac{1}{3}$, are also given.

All simple cases of exchange, factorage, and interest are worked by practice. For example:

At 15 shillings per centum what comes £1008 12*s.* 0*d.* unto?

```
    504  6  0
    252  3  0
   ----------
  7|56   9  0
    20
 ---
 11|29
    12
 ---
    58
   29
 ---
  3|48 | 24 | 12
    100 | 50 | 25    £7 11s. 3 1/2 2/5 d.
```

15*s.* is here considered as ($\frac{1}{2} + \frac{1}{4}$) of a pound. After adding this the division by 100 is done by the vertical dividing line at each step of reduction. The remainder of the pence is reduced to lowest terms.

Two examples will show the Rule of Three with fractions by the commercial method. John Mellis's illustrations show a lack of statement that must have been trying to a learner.

1. If seven pounds of anything cost 3 li. 10 sh., what come 987 pounds to?

```
              li.
   7          3½ ———————— 987
   2                        7
  ---                     ----
  14          7           6909
```

Divide 6909 by 14 to get 493 li. 10*s.*

2. If one pound cost 13 shillings and 4 pence, what will $\frac{7}{8}$ of a pound cost at the same rate?

$$
\begin{array}{ccccc}
1 & \text{———} & 2 & \text{———} & 7 \\
- & & - & & - \\
3 & & 3 & & 8 \\
- & & - & & - \\
8 & & 2 & & 7 \\
24 & & & & 2 \\
\hline
& & & & 14 \\
& & & & 20 \\
\hline
& & & & 280 \text{ sh.}
\end{array}
$$

Divide 280 sh. by 24 to get $11\frac{2}{3}$ shillings.

The Rule of Three is applied in the succeeding chapters to loss and gain without and with deferred payments, length and area problems, exchange, interest, factorship, and barter.

The chapter on tares and allowances is of interest chiefly for the old terms.

Tare is an allowance to the buyer for the weight of the box, barrel, or bag which contains the goods bought.

Treat is an allowance of 4 lb. in every 104 lb. for waste, dust, etc.

Cloff is an allowance, after tare and treat are deducted, that the weight may hold good when sold by retail.

Suttle is the remainder when the allowance of tare only is deducted from the gross weight.

The following problems illustrate the use of these terms:

1. At 16 pound the 100 suttle, what shall 895 pounds suttle be worth, in giving 4 pound weight upon every 100 for Treat?

This becomes:

104 lbs. are worth £16.

∴ 895 lbs. ,, ,, £137 13*s*. $10\frac{2}{13}$*d*.

2. If 100 pounds be worth 36*s*. 8*d*., what shall 860

pounds be worth in rebating 4 pounds upon every 100 for tare and cloff?

First rebate $\frac{4}{100}$ of 860 pounds, or $34\frac{2}{5}$ lb., leaving $825\frac{3}{5}$ lb.

Then, if 100 pounds is worth 36*s.* 8*d.*,
$825\frac{3}{5}$ „ „ „ £15 2*s.* $8\frac{16}{25}$*d.*

3. The question is raised "whether does he lose more that giveth 4 upon the 100, or he that rebateth 4 li. upon 100?"

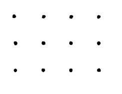
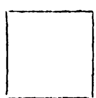
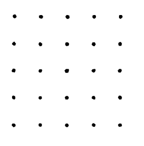
One gives 104 lb. for 100 lb.—*i.e.*, 100 lb. for $96\frac{2}{13}$ lb. losing $3\frac{11}{13}$ per hundred.

The other gives 100 lb. for 96 lb., losing 4 per hundred.

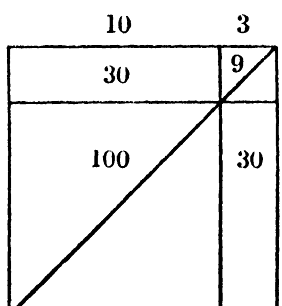
An amount of information is given on the weights used in the towns of Europe with which we deal; but in England itself the weights of London, the Troy and Haberdupoise, are used throughout.

In the sixteenth century the question of interest was coming to the fore. It is not altogether unnoticed in *The Ground of Arts.* In the original dialogue a simple interest is asked as a problem in the rule of proportion composed of five numbers, and in inverse proportion there is a question asking the principal, giving the interest earned, the time, and rate per cent. The scholar evidently understands the matter, as he asks for no explanation. In John Mellis's part of the book there are a few more questions on equivalent loans, it being required to find the time for which they may run, the interest gained, and the rate, but again they always come in as problems in the Rule of Three. Lastly, in R. Hartwell's section there are definitions of interest, simple and compound, and tables of amounts at compound interest, annuities forborne for a number of years, and present money of annuities and reversions. The rate per cent. used is

generally 10, but Hartwell's own table is for 8. These early tables are given in pounds, shillings, and pence, not in decimals of a pound. This is a subject which was much considered and developed in the century following.

There is an appendix by Hartwell treating of roots and squares. He begins by defining a figurate number as one made by the multiplication of one number by another. A parallelogram such as  illustrates the figurate number 20, and a square shown in two ways,  and  is the figurate number 36.

Preliminary to the finding of the square root there is the diagram showing the square 169 made up of



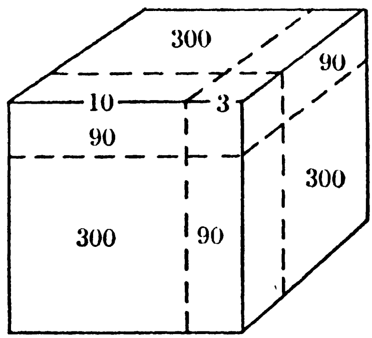
squares of its parts and two rectangles of the product of its parts.

The reader is then guided through the extraction of

the square root with reference to the squares and their complements in this figure.

For solid figurate numbers the cube is drawn and divided into sections, illustrating the theorem (expressed fully in words) that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$



The figure cannot show the cube on the side of 10 units, but marks the other solids. The general extraction of the cube root dependent on this illustration is called the Princely high way.

The inclusion of these chapters in later editions of *The Ground of Arts* probably explains in part why there was only one edition of Recorde's own book on the subject, *The Whetstone of Wit*.

CHAPTER IV

HYLLES, WINGATE, OUGHTRED, AND JAGER

SEVERAL books on arithmetic appeared in the years between 1550 and 1650. Recorde had suggested the order of the topics to be included and their suitable applications, and had given a general idea of what a popular arithmetic should be, and all the new books were on the same lines, but, as was to be expected in those years, with many important additions. As time went on the commercial section became overloaded with tables of weights and measures and examples on allowances, factorage, partnership, loss and gain, and the manifold reckonings that must be made by the man in trade. The section on money was enormously enlarged, and contained tables of exchange between different countries, examples on interest, leases, and annuities, and tables for the ready finding of all that is needed in connexion with compound interest and annuities. On the continent Simon Stevinus had drawn up tables of compound interest using the decimal fractions which gave sufficient accuracy combined with neat display. How he came to think of extending the decimal notation into fractions we do not know, nor if they had long preceded the interest tables. His book on decimals came out in 1585 in Leyden, and was put into English by Robert Norton in 1608 with the title *Disme, the Art of Tenths, or Decimal Arithmetike.* Very soon the tables on interest in our books began to be printed in decimals, and chapters on the fundamental rules of decimal fractions were inserted also.

These years also saw the fancy titles ***Ground of Arts,***

Whetstone of Wit, *Well Spryng of Sciences*, *Pathway to Knowledge*, *Clavis*, and *The Handmaid to Arithmetick* give place to the statement of fact—*Mr Wingate's Arithmetick*, *Moore's Arithmetick*.

The Well Spryng of Sciences, which teacheth the perfect worke and practise of Arithmeticke both in whole numbers and in fractions, by Humphrey Baker, 1562, and later editions, succeeded the *Ground of Arts*. It was a good book, and more modern in the sense that it paid more attention to commercial requirements than had Recorde's original book. Baker gave a fuller chapter to practice, a term used at that time in a wide sense to include any shortened or ready form of reckoning. It had also a chapter on the comparison of weights and measures of different countries. A variant of the crossing-out method of division was used. Baker's book contained an advertisement to the effect that he kept a school on the north side of the Royal Exchange, where he taught various arts, including arithmetic, at a moderate fee.

The Arte of Vulgar Arithmetic, by Thomas Hylles, 1600, is one of the first books to put the rules in rhyme. Hylles claims that "the whole method and metrizing thereof is mine owne and such as no man hitherto hath used, either in English or Latine to my knowledge." Dionis Gray, however, though Hylles did not know it, had used this method in 1577. Hylles tells how he had learned arithmetic and become 'pretty skilfull' in the art. Then, finding that after a lapse of time he had forgotten it, he made rhymes to help himself remember a few of the principal rules. It developed into writing all the rules in rhyme, in which labour he tried to observe six points.

> First that the rules and maxims might contain most certaine and infallible truth: secondly that they might

be prospicuous and intelligible: thirdly that every rule might consist in verses or metres ending with like sounds or terminations, for their better retention in memorie: fourthly that the ceasures of the meetres might fall in apt places: fifthly that every verse might end with a periodus or a colon, or at least with a comma: and sixthly that the rules and maxims might contayne as few words as were possible to declare their proper effect and meaning.

This passage is typical. Hylles says all he has to say and does not mind saying it twice. Hence his rather bulky book.

He consulted many works, among them Tonstall, and gave them due acknowledgment. A tribute to his research is in one of the several laudatory poems at the beginning of the book.

With passing paines, and labor great, the Ants within their Hils
Do hoard up corne as in a barne, to serve them at their wils,
The busie Bees on buds of trees, on herbes and every flower,
Do seeke, and search, and suck the best, to fill their hive or bower.
The husbandmen which hope to have their corne imbarn'd in season
until that time do seldom rest but toile beyond al reason.
Not much unlike a busie Ant, not much unlike a Bee,
Not much unlike a husbandman, this author seems to mee.
For so hath he from booke to booke, and authors many one,
With toil pack't here such dainty corne as elsewhere there is none.

The plan of the book is to write the whole account of arithmetic in metre. It is given one thought at a time by Philomathes, who then enlarges upon, explains, and illustrates the verses. Eumathes, the other interlocutor (for the book is in dialogue), like Recorde's scholar, takes the part of intelligent and

grateful pupil. For example, on the topic of multiplication there are twenty-one pages written upon these verses:

33. Multiplication bends all his devotion
By folding[1] together 2 numbers assignde
To bring forth a third in such like proportion,
To one of those 2 as the other behinde
Is unto an ace, which number we call,
The product or ofcum, wherever it fall.

34. Multiplication first maketh bequest,
To have each number in order so placed,
As in subtraction before was exprest,
Figure to figure with line underplaced
That is to witt, first the multiplicand,
And then multiplier beneath it to stand.

35. Then cause the digit in first place belowe,
To multiplie every digit above,
Orderly placing the products that grow,
Under the line as art doth approve,
Whereof if any a digit surmount
Subscribe the digit, the rest in minde count,
To be transported unto the next place,
As by former rules, you may understand,
And when this first digit hath run his full race,
And multiplied all the multiplicand
The second digit must lead the like daunce,
And so each other as falls to his chance
Si: in placing whole sums you must not forget,
That every digit of the multiplier,
Have the first figure of his product set,
Right under it selfe as straight as a wyer,
And all other figures that thereof accrue,
To left handward thence, each in his place due.

36. And when you have finisht Multiplication,
By every digit of the Multiplier
Draw a new line that next operation,
May show the product in one sum entire,
Which thing by addition is easily wrought,
So have you the ofcum, or product you sought.

37. Multiplication by this means is tried,
First cast out all nines, from the multiplicand,
Setting the rest at such a crosses side $\times$[2]

[1] Compare threefold, manifold.
[2] This cross is an illustration, and not part of the verse.

Which done next take your multiplier in hand,
From whom when all nines be cast out likewise,
Repose the rest at the other side the $\times$[1]
Then folding both sides, from that which doth rise,
All nines likewise, taken the remnant engrosse,
At top of the crosse, and there let it rest,
Till lastly all nines cast out in like sorte,
From the whole product the remnant be prest,
Quite under the crosse the same to supporte,
This done if the top be now like the tayle,
The worke is wel wrought without any fayle.

There are six more short verses on properties of products, such as $a \times b = b \times a$ and if a, b, c, and d are in true proportion then $a \times d = b \times c$.

The intricacies of division occupy thirty-nine pages and need thirty-eight verses. Eumathes finds one rule especially difficult, and asks for an explanation. The verse in question raises and dashes a hope of a new arithmetical symbol.

First make a long sword in what place thou wilt
And pitch down the point, in forme as insues;
Neere to whose handle upon the left hylt,
Place the division your mind is to use.
But let the right hilt have an ace on it made,
And all other digits along the blade. . . .

The 'long sword' is disappointingly prosaic in its use. Hylles is only building up around it the first nine multiples of the divisor, which was often done equally well without guiding lines.

234	1
468	2
702	3
936	4
1170	5
1404	6
1638	7
1872	8
2106	9

[1] This cross is read as 'cross' to rhyme with 'engrosse.'

2 / / : 2

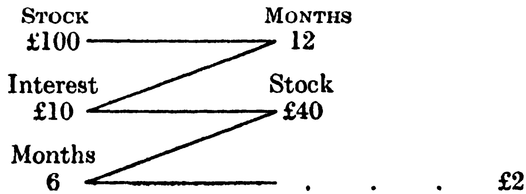
The verses are intended to be learnt by heart. Hylles seriously thinks that they are likely to be remembered when the skill in the rules which comes of practice may be lost. Verse which runs smoothly might, but let us understand his difficulty. He writes:

> For this I dare assure you that he which shall attempt the like methode in any other mathematicall science, shall finde it more easier to make a thousand wanton amorous verses, then ten good playne staves of meeter in that science.

Book I explains the fundamental rules in whole numbers and vulgar fractions. Book II is on arithmetical and geometrical progression, proportion, the Golden Rule and its special uses in partnership, alligation, and single and double position. Much of the early chapters is based on Boetius, Euclid, Tonstall, and Recorde's *Whetstone of Wit.* There are parts of this book which Hylles claims to be his own. They are the rules for dealing with problems in which occur three quantities in continued proportion, and the new and shorter ways of stating the compound Rules of Three (direct, converse, descending, and ascending) in fractions as well as in integers.

An example in Compound Rule of Three Direct in integers is:

> If £100 Stock in 12 months bring £10 interest, what shall £40 bring in 6 months?



Rule. Multiply 10 by 40 by 6 for dividend, 2400.
Multiply 100 by 12 for divisor, 1200.
Divide to get £2 interest.

An example in Compound Rule of Three Converse in Vulgar Fractions is:

> When hay is solde for 20 shillings the loade, the halfepenny bottle weighs 3 pounds, what shall the like bottle weighe when the loade of hay is woorthe but 13 shillings 4 pence?

Shillings	20	3 lb.
	½*d.*	$\frac{40}{3}$ shillings
	½*d.*	. . . 4½ lb.

Rule. Ignore the match terms. Multiply 20 by 3 for dividend, and divide by $\frac{40}{3}$, to find 4½ pounds.

The definition of 'match terms,' a name which appears to be peculiar to Hylles, is given in the verse:

> In questions of proportion discontinuall,
> most commonly, 3 terms are first assigned
> To whom is found a 4 proportionall
> which being found they are by course of kinde,
> Two of one Sirname, and 2 of another,
> So matching by couples like brother and brother.

The match terms in the above example are ignored because they are identical.

The third book gives the shortened workings of the merchants in practice, loss and gain per cent., factorage, bartering, and exchange. Here again Hylles says he is the first to give the Merchant's Rule or Practice. It is in John Mellis's part of Recorde, but not in the neat form given by Hylles. There is no doubt that his claim refers to the arrangement of

work, for he knew both of Recorde's books and also Mellis's additional chapters in the first.

Hylles arrangement is more modern, but he omits the aliquot parts and all explanation of his working. An example is:

> What is the price of numero. 145 hogsheads of oyle at £5 12s. 6d. the hogshead?

725£	0*s*.	0*d*.
72	10	0
18	2	6
815	12	6

Aliquot parts of the shilling and penny are given in these verses:

> A farthing first findes fortie eight,
> An halfepenny hopes for twentie foure,
> Three farthings seekes out 16 streight,
> A peny puls a dozen lower,
> Dicke dandiprat drewe 8 out deade,
> Two pence tooke 6 and went his way,
> Tom trip and goe with 4 is fled,
> But goodman grote on 3 doth stay,
> A testerne onely 2 doth take
> Moe parts a shilling cannot make.
>
> A peny parted by this lore,
> One myte divides by twentie fower,
> Two mytes by 12 and 12 by twayne,
> Three mites by 8 and back again,
> Four mites by 6 and 6 by 4,
> A peny hath not one part more.

All the rules in Book III, except for practice, are worked by Rule of Three.

There are 270 leaves, or 540 pages, in the book, and it ends with an index. The book abounds in examples, for, as Hylles says, "the varietie of examples is one of the chiefest lights in learning," and in spite of redundant words and halting verse it is a good and sound work.

Another good book of this period and one of the very best for many years after was Wingate's Arithmetic. In 1629 Mr Wingate published an arithmetic in two parts, ordinary and logarithmic. When a second edition of Part I was needed he was advised that more rules could be added with advantage. Unable to spare the time himself he invited John Kersey's assistance. John Kersey did the work well, blending his style of writing with Wingate's and at the same time showing clearly in the index what was the original text and what the additions.

In the course of years the book went through several editions, one late edition being dated 1713. The book to be discussed here is described on the title-page as: *Mr Wingate's Arithmetick containing a plain and familiar method for attaining the knowledge and practice of Common Arithmetick. First composed by Edmund Wingate late of Grayes Inn Esq. Fifth edition* 1670. *Revised by John Kersey, Teacher of the Mathematics at the Sign of the Globe in Shandois Street in Covent Garden.*

This is a book to be admired for its direct style. The explanations are sufficiently given in simple language without waste of words. The illustrations are well set out, with explanatory remarks beside each step. Wherever tables occur they are well displayed. The plan is to define the topic first, with an example if it is of benefit, then to take an example and work it through with full directions. The other features of the book are the use of modern division, the complete absence of cancelling, a certain use of algebraical method, and the inclusion of exercises for practice in the rules.

The first fifteen chapters are Wingate's. They follow Recorde in all the rules up to falsehood, with

the exception of fractions, which are to be found in the next sixteen chapters by Kersey. Extraction of roots and proportion by Wingate, and an appendix on commercial methods by Kersey complete the book. In the actual teaching of the topics the differences from Recorde mostly show advancement.

Recorde's terms *ternaries* and *trinities* for the sets of threes into which long numbers are divided give place to *periods* with *sirnames*—*e.g.*, five hundred and twenty-one *millions*, four hundred and twenty-six *thousands*, three hundred and forty-one. Many tables are given, among them the dozens.

> A gross is 12 dozen.
> A dozen is 12 particulars.

The table of time is followed by

> Thirty days hath September, April, June, and November,
> February hath twenty-eight alone, and each of the rest
> thirty-one.

Wingate teaches long division because it

> is to be preferred before any of the common wayes of dividing by dashing out of figures, where the steps of the division are so confounded (besides the burden upon the memory by a promiscuous multiplication and division) that if any error happen, it can scarce be corrected without beginning the work anew.

His rule for the process is:

> First you must ask how oft, in Quotient answer make;
> Then multiply, subtract, a new dividual take.

A dividual is a partial dividend. There is no crossing out of figures, but a dot is put under each figure of the dividend as it is used. Long division is used even for dividing by digits.

In dividing by a power of 10 the vertical, without the sub-horizontal, line is used.

The Rule of Three is still a rule to be memorized, but the worker is recommended to decide by the sense and tenor of the question whether it falls under the rule direct or inverse.

To come to Kersey's work, division in vulgar fractions is not so clearly arranged as in Recorde.

$$7\tfrac{1}{2})\ 42\ ($$

$$\left.\frac{15}{2}\right)\frac{42}{1}\left(\frac{84}{15}\right.$$

$$15)\ 84\ (5\tfrac{3}{5}$$

Decimal fractions was a new branch of the subject. John Kersey sees them as invaluable in tables. Otherwise, he says, there are times when practice by aliquot parts will bring out an answer more speedily. Later he says:

> If the greatest integer of money, as also of weight, measure, etc., were subdivided decimally . . . the doctrine of Arithmetic would be taught with much more ease and expedition than it is now; but, it being improbable that such a reformation will ever be brought to pass, I should proceed in directing a course to the studious for obtaining the frugal use of such decimal fractions as are in his power.

The French Revolution had not then taken place.

The point, the comma, and the vertical line are mentioned in notation, but Kersey himself uses only the point or line. Addition and subtraction proceed as in whole numbers. In multiplication and division the numbers are treated as whole numbers, and then the decimal point is put in by counting places in multiplication and by inspection in division. The rule for decimalizing money at sight is given. After showing how to work out decimals of £1 he goes to the somewhat unprofitable labour of giving tables of the

value in terms of pounds to seven places of decimals of every number of farthings up to a shilling and every number of shillings up to a pound. Similar exhaustive tables in Troy and avoirdupois, liquid and dry methods, long measure, dozens, and time are also given.

Extraction of roots is not well treated. A diagram of 5 units by 5 units divided into its 25 component squares illustrates the meaning of square and square root. A die of little dice illustrates cube and cube root. The rules for extracting square and cube roots, however, are given without reason either by a diagram or the algebraical identity. This chapter is by Wingate, with revision by Kersey.

The last two chapters are Wingate's. The first one is Boetian in character. It is a discourse on the relations which exist between two numbers either in difference, or in rate or 'reason,' and contains a mass of explanations of the many relations between antecedent and consequent. This is preliminary to a treatment of arithmetical and geometrical progression and proportion used in Rule of Three. The rules for sum and relations of means and extremes are given in words. He does not appear to be able to find the first term or difference except of an arithmetical series in which they are equal, for instance, in the series 2, 4, 6, 8, . . . and 3, 6, 9, . . . , when you divide the last term by the number of terms.

The appendix, a considerable part of the book, is the work of Kersey. It has many points of interest; the meanings of signs, the strong section on interest, business abbreviations, the use of a letter in solving a problem, and the problems and parlour tricks at the end.

The earlier books used very few signs. Kersey's

complete list is therefore valuable. $+$ means 'added to' also 'not completely expressed'—*e.g.*, 1·414$+$. There are $-$, $\times$, the horizontal line for bracketing, $\square$ for rectangle, the dividing line in fractions, : :: : for proportionality, $=$ for equality, $<$, $>$, $\sqrt{}q$ for square root, $\sqrt{}c$ for cube root. The only mark for division is ··)····(··.

Among the suggestions for speedy reckoning are these. The multiplication tables, including 12 times 1, 2, 3, . . . 9, should be learnt by heart for their usefulness in money sums. Division can sometimes be done by factors—*e.g.*, 3466 farthings to shillings.

8)3466
6) 433(72 sh. $2\frac{1}{4}d.$

Factors may be cast out in working Rule of Three. Practice by aliquot parts may replace Rule of Three.

Very considerable space is given to exchange, and there are full tables connecting money, weights, and measures of several countries. In the problems on proportion, fellowship, alligation, etc., he frequently uses the letters A, B, C, D, etc., to save writing in words and connects them with the signs $+$, $-$, $\times$ until the work looks like algebra. Thus, to find a fourth proportional he writes:

$$24 : 6 :: 36 : Q$$
$$24 \times Q = 6 \times 36$$
$$Q = \frac{6 \times 36}{24}$$

Alligation was much used in medicine, and a typical problem is:

> Suppose a medicine to be compounded of several simples, whose quantities and qualities are as followeth, viz.: 4 ounces of a simple which is cold in 2° and moist in 1°, 5 ounces hot in 3° and (in respect of dryness and

moisture) temperate; 3 ounces hot in 2° and dry in 2°, 6 ounces hot in 1° and moist in 4°, 4 ounces cold in 3° and moist in 2°. The question is to know the temper resulting?

The subject of interest, growing in importance through the century, covers in this book over seventy pages. Simple interest is not reasoned by rule of three, but, though stated fully in words, taught by the formula $\frac{Prt}{100}$. Rebate is taught by Rule of Three, and compound interest by continued proportion. There are a few tables for present worth in simple interest, but the tables in compound interest are a tribute to the work in decimal fractions of Stevinus. In them the term of years is 30, the rate of interest 4, 5, 6, . . . 12 per cent., and the tables are for the amount of £1 at compound interest, the amount of a £1 annuity, the present worth of £1, the present worth of a £1 annuity, and what annuity £1 will purchase. Leases and fines for renewing leases are explained. Land on lease appears to be held mainly from cathedral churches and colleges.

The thirteenth edition of Wingate, edited in 1713 by George Shelley, writing master at Christ's Hospital, makes some additions in the supplement. There are hints for contracting work—*e.g.*, dividing by 112 in two lines—and the use of factors. Flemish exchanges are included and also a section on practical mensuration. A pleasing phrase is preserved in the note that tare and tret and other allowances are "call'd beyond seas *the Courtesies of London*, because not practised in any other place."

Several editions of *Clavis Mathematicæ*, by Oughtred, appeared between 1631 and 1702, two in English

as *The Key of the Mathematicks.* The book described here is dated 1667. It was a book on mathematics of some celebrity, a scholarly work to which reference is frequently made. It cannot, however, be considered with Recorde, Baker, Hylles, and Wingate as a first book or a popular book on arithmetic. Its language and, even more, its brevity, are against it.

Whole numbers and decimal numbers are taught simultaneously, and this necessitates the extension into decimal fractions being taught with the Arabic notation. New terms are introduced. Positive, +, or increasing numbers are those spreading from the units place to the left, and negative, −, or decreasing numbers are those spreading to the right, going down in powers of tenths as the others rose by powers of ten. The words positive and negative here have not any directional sense, nor have the signs + and −. Later on in the book the signs are used in their other meaning of 'add to' and 'take away' from.

Throughout the book Oughtred retains the sub-horizontal line with the vertical to separate the fractional numbers, the symbol which Recorde used when dividing by the powers of ten.

Addition and subtraction are completed in three sentences apiece, with one illustration in whole numbers, one in decimals, and one in money. Consequently the carried figure in addition and the added tens in subtraction go entirely unexplained. The right-hand digit of the multiplier is used first in both full and contracted multiplication.

To illustrate division, 187135075 is divided by 297 by two different methods, one, A, a modification of old division and the other, B, an early arrangement of the new (long) division. In old division the divisor was usually placed below the dividend and moved a

place to the right at each step of the working. In A the successive multiples of the divisor, which were not usually shown, are placed below the dividend.

A.

$$\begin{array}{l}
\qquad\qquad 12 \\
\qquad\ \ 8921317 \\
297)187135075(630084\tfrac{127}{297} \\
\qquad 178213768 \\
\qquad\ \ 892118
\end{array}$$

B.

$$\begin{array}{l}
187135075(630084\tfrac{127}{297} \\
\ 297 \\
1782 \\
\quad 893 \\
\quad 297 \\
\quad 891 \\
\qquad 2507 \\
\qquad 297 \\
\qquad 2376 \\
\qquad\ \ 1315 \\
\qquad\quad 297 \\
\qquad\ \ 1188 \\
\qquad\quad 127
\end{array}$$

In the chapter on powers and roots there is a table of powers of numbers up to 9^9, and of powers of the quantity A + E, using the old notation—*e.g.*, Aq + 2AE + Eq and Ac + 3AqE + 3AEq + Ec.

The steps in the extractions of roots are explained with reference to these formulæ instead of to the diagrams of the dissected square and cube of Recorde.

There were a few other books on general arithmetic published in the century 1560 to 1660: Thomas Masterson's group of four books which came out between 1592 and 1595; Moore's *Arithmetick*, 1650, abridged in 1681 for Christ's Hospital; Noah Bridges' *Vulgar Arithmetique*, 1653 and 1661, teaching modern

long division; and Nicholas Hunt's book of 1633, with the descriptive title:

> *The Handmaid to Arithmetick* refined: showing the variety and facility of working all Rules in whole numbers and fractions, after most pleasant and profitable waies. Abounding with Tables above 150 for monies, measures and weights, tale and number of things here and in forraigne parts; verie usefull for all Gentlemen, Captaines, Gunners, Shopkeepers, Artificers, and Negotiators of all sorts; Rules for commutation and exchanges for merchants and their Factors. A Table from £1 to 100 thousand for proportionall expences and to reserve for Purchases.

There were also books dealing specially with interest or leases and annuities, such as those by Richard Witte, 1613, and William Webster, 1634.

Robert Jager's *Artificial Arithmetic in Decimals*, 1651, is not mentioned in Solomon Lowe's list of writers at the end of his *Arithmetic*, 1749, or described by either Peacock or De Morgan.

The book does not follow the ordinary course. In the foreword to the reader Jager proposes to do away with the intricate and tedious reductions of vulgar fractions and to work everything by whole numbers (decimals) whether the problem be in money, weight, or measure, or in the rules of interest, barter, exchange, fellowship, etc. It is intended to be of use to merchants, tradesmen, artificers, gunners, and practical geometricians.

The first forty-seven pages contain numerous examples in addition, multiplication, and division of decimals, and clear directions for fixing the 'prime line,' or the line between the whole numbers and decimal fractions. Then, after presupposing through all this work a knowledge of the first four rules, Jager

begins on page 48 to explain the Hindu-Arabic notation and rules of reckoning. Here, even before division is taught, he uses it to prove multiplication. This faulty arrangement is a blemish on an otherwise very good and suggestive book. Jager is riding a hobby in making decimals oust all other fractions, but nevertheless he explains simply and clearly, making one see decimals not set apart as difficult matter for more mature workers, but as falling naturally into place in an extension of our notation, and requiring no more than a few directions peculiar to their needs as the money reckonings do. This is the view of a man writing when decimal fractions had not long been known, and there was not even agreement as to how to write them.

This book seems to have been forgotten, and it may be that this 1651 edition was the first and last. Probably the entire omission of vulgar fractions affected its popularity. Vulgar fractions have their uses. $\frac{1}{3}$ and $\frac{1}{7}$ are fractions more easily expressed and more easily conceived than 333333 and 142857, and men following ordinary occupations prefer brevity in work and statement. A thresher, then as now, would not welcome in his accounts such an entry for the cost of barley as £1|89716125 even if he was able to convert it at sight into shillings and pence, nor would a merchant care to see 418|763632 pounds as the price of "a parcell of cloves." There is also the date of publication of the book to consider, 1651. Cocker appeared in 1677, and for some unexplained reason crowded out all the other better books.

According to Jager the decimal numbers are called primes, seconds, thirds, etc., as in the old sexagesimal fractions. The notation is 2598, and if there is a whole number the 'prime line' also is employed—*e.g.*, 9|351.

There is a money table expressing $\frac{1}{4}d.$ to $11\frac{3}{4}d.$ and $1s.$ to $19s.$ as decimals of a pound. $\frac{1}{4}d.$ is $\dot{0}\dot{0}\dot{1}\dot{0}\dot{4}\dot{1}\dot{7}$. This table was intended to be learnt by heart.

The early pages on the rules of decimals are the most important. Directions are given for halving decimal numbers, adding numbers which are partly whole and partly fractional, and reducing a vulgar fraction to a decimal by dividing numerator by denominator, a rule which does not present difficulty because the whole of the quotient is in the decimal places. There is then shown the method of finding by multiplication the value of a decimal of a pound, a month, a hundredweight, an hour, and the way of evaluating a decimal of a pound money at sight with the adjustment for farthings. Where most books deal with very simple cases of vulgar fractions, Jager takes a long 'fraction of fractions' to bring to a decimal, namely,

$\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$ of $\frac{7}{8}$ of $\frac{8}{9}$ of $\frac{9}{10}$ of a pound.

He steadily puts together all the numerators and all the denominators by multiplication, and then divides out to find $\dot{1}\dot{0}$ of a pound. There is no cancelling and the division of 362,880 by 3,628,800 has to be performed. There are several cases of reduction where division is not complete, but the fact that some quotients have recurring figures goes unnoticed. The prime line is placed in a product by adding the number of places after the prime lines in the multiplicand and multiplier. Multiplication of a decimal number by a power of ten is performed in two steps. First, ignoring the decimal point, the number is multiplied by the proposed power of ten by the addition of the requisite number of ciphers. Then the prime line is so placed that it is followed by as many figures or ciphers, for

ciphers may in this instance be regarded as figures, as there were figures after the prime line in the multiplicand.

The placing of the prime line in a division is a more difficult matter, and several special cases are considered. Wherever there are whole numbers to guide the position of the prime line is determined by inspection. Old and new forms of division are used side by side in this book, but in the three illustrations of division by a whole number, or by a number partly whole and partly fractional, the new long division is used and the process is easy to follow. The points above the figures mark the fractional part of the number; the points below are added one by one as the figures are brought into use.

```
(i)        33)1666|03750(50|48598
              165
              ---
                160
                132
                ---
                   etc.

(ii)      243)63|78750(0|2625
              48 6
              ----
              15 18
              14 58
              -----
                  etc.

(iii)      16|5)15675|00(950
                1485
                ----
                 825
                 825
                 ---
                 000
```

The difficulties begin with the fractional divisors—*e.g.*, 17 to be divided by $\dot{0}\dot{3}\dot{7}\dot{5}$. The rule is, observe

what is the denomination of the first figure of the divisor. The first significant figure is 3, which is a 'seconds,' that is, a figure in the hundredths place. Build up the whole number to 'seconds,' namely, 170̇0̇. Then the first figure of the quotient will be hundreds (found by inspection of 1700 and 3). The working is:

```
                1 1|1
              1 2 2 2|2
          2 1 2 5 5 5 5|5
0 3 7 5)1 7 0 0|0 0 0 0 0(4 5 3|3 3 3
          3 7 5 5 5 5 5 5
            3 7 7 7 7 7
              3 3 3 3
```

The last case, the division of a fraction by a fraction, is argued from this rule.

> Although by adding one or more cyphers, or fractions, to the whole numbers, you turne the whole numbers into the same denomination that the first figure of the divisor is off. And although you cut off one or more fractions with a prime line, yet the cyphers, and Fractions so added, and cut off, are esteemed for whole numbers, and produce whole numbers in the quotient, as is before declared. And therefore looke how oft you can find the first figure of the divisor in the whole numbers of the dividend. . . .

In strange guise we recognize the modern 'standard form.'

Numeration and the ordinary arithmetical rules begin on page 48. A good feature of the book is that after the definition it passes as quickly as possible to an example, and in the explanation of this the rule emerges.

In all the cases of the Golden Rule Jager arranges the terms around the symbol Z in the same order

as Recorde, but in reading the proportions across the lines instead of downward he states relations which are outside the modern definition of ratio:

As the fewer Els of Holland cloth to the price thereof
So the greater number to the price thereof

or *As* the fewer men to the first time
So the greater number of men to the second time.

It is a step forward to try to state a proportion, for hitherto the terms were set about the symbol, and multiplication and division went by rote.

There is an interesting range of problems on the Rule of Three, diameter, circumference, and areas of circles, taking height by length of shadow, price of cloth, interest on money, amount of wine for a number of men, oats for a number of horses, cloth for hanging a room, and other matters outside commerce.

"The Rule of Practice by Multiplication only" is intended for such examples as: Given the cost of one yard, what will another length cost? The price and the given length are put into decimals, and the statement made about the Rule of Three symbol, but as the first term is unity there is need only to multiply the other two terms together.

"The Rule of Practice by Division only" is used for such examples as: (*a*) Having the price of one pound weight of anything, find what quantity any other sum would buy. (*b*) Having the price of, say, 1000 yds. of material, find what one would cost.

The Rule of Proportion differs from the Rule of Three because in it if three numbers are given there can be found a fourth, a fifth, and others in continual proportion. Thus he ties down the word proportion to a series. The rule is used for finding compound interest.

The Rules of Fellowship, Loss and Gain, and Exchange are special cases of the Rule of Three, and are taught by the 'as and so' of that rule.

> *As* the whole adventure is to the whole gain
> *So* is every man's perticular adventure to every man's perticular gain.

And

> *As* the totall of the goods laden aboard to the totall of the goods cast overboard,
> *So* is every merchants perticular goods to every merchant's perticular losse.

Equation of payments covers a wider field than is usual. First is given an ordinary example:

> A merchant hath owing him 655 pounds, to be paid 200 pounds present, 100 pounds at 5 moneths, 300 pounds at 9 moneths and 55 pounds at 12 moneths.
> Now multiply 100 by 5, makes 500,
> then multiply 300 by 9, makes 2700,
> and 55 multiplied by 12, makes 650,[1]
> all of which products added together make 3850,
> which divided by 655, the whole money owing,
> the quotient will be 5|87786 which is 5 moneths and 26 days, and at such time it ought to be paid at one payment.

A later example is:

> Four men have **300** sheep, whereof the first hath **100**, the second **40**, the third **150**, and the fourth **10**. They give unto a shepheard to keep these sheep **25** pound wages a year, meat, and drinke: The question is what part of the wages every man must pay, and how many days every man must dyet him?

Unusual in this chapter on equation of payments is the problem of the twins.

Jager says very little about progressions, but he shows how to find the sum of simple arithmetical series.

[1] This mistake is carried through to the end of the working.

The example in the Rule of Falsehood, single position, is this:

> A man passing by a maid, keeping a flock of geese, said: God speed fair maid and your 20 geese. No, sir, said she, I have not 20, but if I had as many more, and half as many more, and two geese and a half, then had I twenty. I demand how many Geese she had?
>
> Suppose 9, then as many more would make 18, and half as many more make 22|5, and 2 geese and a half make 25; now if 25 come of my supposition 9, what will come of 20?
>
> As 25 to 9; so are 20 to 7
>
> and so many geese the maid had.

The above example is typical of the simpler problems in falsehood, which could be solved by 'single position,' and in which only one experimental solution was made. The proposal that the girl has 9 geese is made in the expectation that when, in accordance with the conditions of the problem, there are added as many more geese, and half as many, and two and a half, the total will prove to be a number of geese other than 20. The total comes, in fact, to 25. Then by Rule of Three 'as 25 to 9; so are 20 to 7 (or 7|2)' and the girl is found to have 7 geese. The mistake in the wording of the question standing immediately before the proportion could be rectified by saying 'what will produce 20?'

An example in double position is worked in Chapter V under the Rule of Falsehood by double position. The book ends with a few simple cases of finding distances, etc., by sighting with instruments and working out the results by similarity of triangles.

CHAPTER V

COCKER

COCKER'S *Arithmetick* appeared first in 1677, and was still used in the first quarter of the nineteenth century. Like Recorde's *Ground of Arts* it was, from the point of view of long popularity, one of the chief arithmetics in England before modern times.

De Morgan wrote at greater length on Cocker than on any other book, and the six pages he devoted to it are all in condemnation. He believed it to be a forgery, and gave his reasons for thinking that John Hawkins, who professed to edit it, had in reality written it, possibly with some help from Cocker's papers, and published it as the work of that eminent teacher. He showed how it was compiled from Recorde, Moore, Bridges, Hodder, and others, although by one who was not master of the subject. "I am of opinion," he wrote, "that a very great deterioration in elementary works on Arithmetic is to be traced from the time at which the book called after Cocker began to prevail."

De Morgan's is a serious indictment of a book which survived and was esteemed for nearly one hundred and fifty years, but he was nearer in time to Cocker than we are, and was an active participator in the movement for more enlightened teaching of arithmetic. Now, nearly ninety years on, we look more calmly on what is to us only history, and, moreover, we know the task of eliminating Cocker's faults to be well in hand.

The comparatively early edition of 1688 is described here. Its title runs:

> Cocker's Arithmetick: being a plain and familiar method, suitable to the meanest capacity, for the full understanding of that incomparable Art, as it is now taught by the ablest schoolmasters in city and country. Composed by Edward Cocker, late Practitioner in the Arts of Writing, Arithmetick and Engraving. Being that so long promised to the world. Perused and published by John Hawkins, Writing Master near St George's Church in Southwark, by the author's correct copy, and commended to the world by many eminent mathematicians and writing masters in and near London.

There are a few pages of prefaces and forewords. They show careless editing at the least, and could scarcely have been repeated in edition after edition had there been our present system of reviewing books.

"Suitable to the meanest capacity." In the early chapters there are references to Hylles, Boetius, Oughtred, Wingate, Euclid, and others. The explanation of numbers contains arguments on roots of numbers, trigonometrical ratios, logarithmic numbers, and the properties of water. Of the subject generally the author writes:

> For as magnitude or greatness is the subject of Geometry, so multitude or number is the subject of Arithmetick, and if so, then their first principles and chief fundamentals must have like definitions or, at least, a semblable congruency.

The reader would be wise to omit the definition and preliminary notes on every rule. The meaning of the maze of words is not clear and emerges only as the chapter advances. Indeed, throughout the book "he draweth out the thread of his verbosity finer than the staple of his argument."

Chap. 12. *of Three Direct.* 203

```
            100——6——75
         A 12    |     9        B
  l.     l.    l.|          then say
If 100   6     75|  m.  l.  s.   m.
         75      |If 12--4--10---9
        ———      |      20
         30      |    ————
         42      |   90 shillings
      ————— l  s |   12
1|00) 4|50 (4·10 |  —————
        4        |  180
   ———           |   90
Remains (50)     |  ——————
Multiply 20      |  1080 pence
      ————       |     9
1|00) 10|00 (10 s.| ———————  12) 2|0) l.  s  d.
                 |12) 9720 (810 (6|7 (3--7--6
                 |
     l.  s.      |   96      72
Facit 4--10      |  ————   ————
                 |   12      90
                 |   12      84
                 |  ————   ————
                 |  (0)     (6) pence
                 |
                           l.  s.  d.
                  Facit    3——7——6
```

So that by the foregoing Operation I conclude that if 100 *l.* in 12 months gain 6 *l.* Interest, 75 *l.* will gain 3 *l.* 7 *s.* 6 *d.* in 9 months, after the same Rate.

The

A Page from Cocker's "Arithmetick," 1688 Edition

A problem in the Double Rule of Three Direct: If 100 *l.* in 12 months gain 6 *l.* interest, what will 75 *l.* gain in 9 months?

In explaining the notation he refers to the old idea of the unit that it is the beginning of number but itself no number. Cocker would have "a cypher to be the beginning [in the previous sentence 'limit'] of number, or rather the medium between increasing and decreasing numbers commonly called abstract or whole numbers and negative or fractional numbers." Thus in one sentence is a contradictory definition of the cipher, and he uses the word 'abstract' where he should have written 'positive.' He is referring to the decimal notation carried through to fractions, as explained in Oughtred. Nowhere in all the paragraphs given to the cipher is there mention of its use in holding an empty place.

Decimal fractions, he says, may be written in three ways, namely, 32·75, $32\underline{|75}$, or $32\frac{75}{100}$. Vulgar fractions have a line separating numerator and denominator, but, with one or two exceptions, there are no lines in the 1688 edition, an omission probably of the printers, since they are mentioned in the text.

Chapter II gives the tables of weights, measures, and money. Chapter III names the species of arithmetic—natural, artificial, analytical, algebraical, lineal, and instrumental. This book treats only of natural arithmetic, which may be positive or negative. Positive again may be single—*i.e.*, the processes in numbers—and comparative—*i.e.*, their applications. Negative arithmetic is the Rule of False Position.

In the chapters on the rules the plan is to give first a definition, then a statement of the rule, illustrations in abstract numbers, money, weights, and measures, a proof, and finally a few examples to show the type of questions which can be answered by the rule.

Subtraction, which never had been allowed a rational explanation, became even less comprehen-

sible on account of Cocker's, or Hawkins', teaching by reason of the loose wording.

```
437503
153827
------
283676
```

> I begin, saying 7 from 3 I cannot but (adding 10 thereto I say) 7 from 13 and there remains 6 which I set under the line in order, then I proceed to the next figure saying 1 that I borrowed and 2 is 3 from 0 I cannot, but 3 from 10 and there remains 7.

Although in the meantime subtraction had been explained, the unfinished verb and the word 'borrowed' persisted until less than fifty years ago, and children were left to worry over its meaning until the cane, the dunce's cap, and the fact that the answer always came right helped the wisest to leave the problem unsolved.

Modern long division is used, the digits of the dividend being pricked off with dots as they are brought down to complete the 'dividuals.' In division by 100 the vertical line only is used as a separator, although in an earlier chapter the sub-horizontal line was added in the decimal notation. The use of long division even by units gives a simple reduction an unusual appearance—*e.g.*, 126720 farthings to pounds.

```
                      12)          2|0       1
4)126720             (31680      (264|0    (132
  12                  24          2
  --                  --          -
   6                  76          6
   4                  72          6
   -                  --          -
   27                  48          4
   24                  48          4
   --                  --          -
    32                 (0)        (0)
    32
    --
    (0)
```

The pleasure in great numbers is seen in the examples; how many barley corns to encircle the earth? and, how many minutes since Christ was born?

The short chapter on progressions shows how to find the sum of terms and gives relations between means and extremes.

In Rule of Three the directions are very explicit and the work goes by rote. After the explanation that two of the terms of the question are of one kind, and the number sought of the same kind as the other term given, the rule proceeds thus:

> Again, observe, that of the three given numbers, those two that are of the same kind, one of them must be the first, and the other the third, and that which is of the same kind with the number sought, must be the second number in the rule of three; and that you may know which of the said numbers to make your first, and which your third, know this, that to one of those two numbers there is always affixed a demand, and that number upon which the demand lieth must always be reckoned the third number.
>
> In the rule of three direct (having placed the numbers as is before directed, the next thing to be done will be to find out the fourth number in proportion, which that you may do) multiply the second number by the third, and divide the product thereof by the first, or (which is all one) multiply the third term (or number) by the second, and divide the product thereof by the first, and the quotient thence arising is the fourth number in a direct proportion, and is the number sought, or answer to the question, and is of the same denomination that the second number is of.

The rules on mixtures are called alligation medial and alternate. Alligation medial is simple, answering such questions as:

> A farmer mingleth 20 bushels of wheat of 5*s.* a bushel with 36 bushels of rye at 3*s.* a bushel and 40 bushels of

barley at 2*s.* a bushel. Now, I desire to know what one bushel of the mixture is worth.

It takes six pages of working to solve a problem in alligation alternate.

Question. A certain farmer is desirous to mix 20 bushels of wheat at 5*s.* or 60*d.* a bushel with rye at 3*s.* or 36*d.* a bushel, and with barley at 2*s.* or 24*d.* per bushel and oats at 1*s.* 6*d.* per bushel, and desireth to mix such a quantity of rye, barley and oats with the 20 bushels of wheat so that the whole composition may be worth 2*s.* 8*d.* or 32*d.* per bushel.

To this problem three solutions are found.

Rule: (*a*) Link an expensive grain with either one or both of the grains of lower value than the mixture.

N.B. The numbers are used abstracted from their denominations in Cocker. In this working they stand first for the number of pence in the price, and in the later columns for the number of bushels. Case III is a combination of Case I and Case II.

Case I	Case II	Case III
$32\left\{\begin{array}{l}60\\36\\24\\18\end{array}\right.$	$32\left\{\begin{array}{l}60\\36\\24\\18\end{array}\right.$	$32\left\{\begin{array}{l}60\\36\\24\\18\end{array}\right.$

(*b*) Put the difference between 32 and each number in turn opposite the number to which it is linked, e.g. 32 ~ 60 is 28, put 28 opposite 18.

Case I		Case II	
$32\left\{\begin{array}{l}60\\36\\24\\18\end{array}\right.$	$\begin{array}{r}14\\8\\4\\28\end{array}$	$32\left\{\begin{array}{l}60\\36\\24\\18\end{array}\right.$	$\begin{array}{r}8\\14\\28\\4\end{array}$

Thus we get two mixtures of required consistency.

CASE I. 14 bushels of wheat with 8 of rye, 4 of barley, 28 of oats.

CASE II. 8 bushels of wheat with 14 of rye, 28 of barley, 4 of oats.

The question stated that there were 20 bushels of wheat, therefore the corresponding quantities of the other grains must be found by repeated applications of the rule of three.

CASE III. Apply rule (*b*) twice, because each number is connected with two others. Add the resulting numbers.

CASE III

32	60	8	14	22
	36	8	14	22
	24	28	4	32
	18	28	4	32

22 bushels of wheat must be put with 22 of rye, 32 of barley, 32 of oats. By Rule of Three find the quantities to go with 20 bushels of wheat.

The treatment of vulgar fractions is very simple. Rule of Three is then repeated, introducing fractions.

More attention is paid to commercial problems, ready reckoning, practice, barter, loss and gain, equation of payments, exchange of moneys, and weights all featuring.

The Rule of False Position is divided into single and double position. The example in single position is:

> A person having about him a certain number of crowns said if the fourth and the third and the sixth of them were added together, they would make just 45. Now I demand the number of crowns he had about him.
>
> Suppose 24, of which the fourth is 6, the third is 8, the sixth 4, in all 18. By rule of three, as 18 is to 24 so is 45 to the required number, which is 60.

There are two pages of explanation of double position, ending with the verse:

> When Errors are of unlike Kinds
> Addition doth ensue,
> But if alike, Subtraction finds
> Dividing work for you.

Question. A, B, and C build a house which cost 76 l. of which A paid a certain sum unknown, B paid as much as A and 10 l. over, and C paid as much as A and B, now I desire to know each man's share.

Having made a cross according to the second rule, I come according to the third rule to make choice of my first position, and here I suppose A paid 6 l. which I put upon the cross you see, then B paid 16 l. (For it is said he paid 10 l. more than A) and C paid 22 l. for 'tis said he paid as much as A and B, then I add their parts and they amount to 44, but it is said they paid 76 l.,

l.					l.
9				A	6
19	120		288	B	16
28	6	168	9	C	22
56	12)		(14	Sum	44
76					76
56	32	12	20		44
20				Error	32

wherefore it is 32 too little, which I note down at the bottom of the cross under its position for the first error. Secondly I suppose A paid 9 l. then B paid 19 l. and C 28 l., all of which added together make 56, but they should make 76, wherefore the error of this position is 20, which I put at the bottom of the cross under his position for the second error, then I multiply the errors and the positions crosswise, viz.: 32 (the error of the First position) to 9 (the second position) and the product is 288. Then I multiply 20 (the error of the second position) by 6 (the first position) and the product is 120.

Then (according to Rule 4) I subtract the lesser product from the greater, viz. 120 from 288, because the

errors are both alike (viz.: too little) and there remaineth 168 for a dividend. Then I subtract 20 (the lesser error) from 32 (the greater error) and the remainder is 12 for a divisor, then divide 168 by 12 and the quotient is 14 for the answer which is the share of A in the payment.

If it were done, when 'tis done, then 'twere well it were done quickly. Arithmetic awaited 'x.'

For fuller explanation this example is worked also for the case when both the errors are too large and again with one error too large and one too small. In the latter case after the cross multiplication, addition instead of subtraction gives both dividend and divisor.

The St Andrew's cross in this example is the only sign in the book. It is used to connect sets of numbers to be multiplied, though Oughtred by this time had tried to pin it down to the meaning 'multiplied by.'

The book ends with "*Laus Deo Soli.*"

The fifty-third edition published in 1750 shows very little change in either matter or treatment. It has the same chapters, and the same examples in almost identical words. There are not so many references to previous writers, and the rhyme in the last chapter is omitted. It is a smaller book printed in modern type, and several mistakes of the earlier books are corrected.

This was a book very much criticized during the reforms of the nineteenth century. The reformers found that they had to take in hand much more than the mere revision of the rules, and in the matter of poor teaching Cocker, or Hawkins, was the greatest offender. John Hawkins talks over the pupil's head, he is master of high-sounding phrases which frequently in explanations show a lack of precision which plays havoc with the units, and throughout directions for working are given without reasons. His *Arithmetic* is an argument for the proposition that, given that

the rule is intelligently evolved beforehand and stated to him in sufficient detail, the learner can arrive at a result without knowing in the least what he is about.

Decimal fractions, logarithms, and algebra are the three parts of the second book, Cocker's *Decimal Arithmetick*, 1685. The sixth edition, 1729, has nine pages of forewords—John Hawkins' stupendous opinion of his own work, three laudatory poems to Cocker and two to Hawkins—before the business of the book begins.

The alternative ways of writing decimal fractions are given.

> A decimal fraction being written without its denominator is known from a whole number by having a point or prick prefixed before it, thus ·25 is $\frac{25}{100}$ . . . But some authors distinguish decimals from whole numbers, by prefixing a VIRGULA, or perpendicular line, before the decimal (whether it be alone or joined with a whole number) thus |8 is $\frac{8}{10}$ and |025 is $\frac{25}{1000}$; and 29|16 is $29\frac{16}{100}$ etc. Others express the same decimal fractions and mixed numbers thus, viz.: $|\underline{8}$ $\quad |\underline{025}$ $\quad 29|\underline{16}$ etc., others with a point over the place of units in the whole number, and then the former fractions and mixed numbers will be thus written, viz: 8,025 29,16[1] and the like of others. And some authors again put points over all the places or figures in a decimal fraction thus $8{,}\dot{0}\dot{2}\dot{5}$ $\quad 29{,}\dot{1}\dot{6}$ $\quad 48{,}\dot{0}\dot{2}\dot{5}$ etc., but, being written according to the first direction, I conceive they may be most fit for calculation.

Thus Cocker has done us the double service of collecting together most of the methods known of separating decimal fractions from whole numbers, and of enforcing the use of the clearest of them, the decimal dot or point (·). There is little more to add to his

[1] Cocker writes *a point over* the unit, but shows in the examples a *comma after*. The comma after the unit was frequently used, whereas a point over the unit was practically unknown. In the first example the 8 is a unit digit before the decimal fraction ,025.

account. Simon Stevinus, 1585, in *La Disme,* wrote decimals in the form $544'2''3'''8''''$, and for many years the decimal numbers were quite as often known as primes, seconds, thirds, as tenths, hundredths, thousandths. Napier, almost his contemporary, had no settled practice. Sometimes he used the comma, sometimes the virgula or prime line, and sometimes the virgula combined with Stevinus's marks for primes, seconds, thirds. Richard Witte in his interest tables, 1613, used the virgula, Briggs the sub-horizontal line, and Oughtred the two combined. Gunter, like Napier, experimented, using sometimes the lower line, sometimes this line combined with the point (·), and in the great part of his work the point only. In the course of years the point emerged as the favourite symbol, and then Cocker's *Decimal Arithmetick* insisted on its use.

In Cocker the treatment of decimals is quite elementary. The value of $\frac{9}{11}$ is worked out. It is ·818, or a place further ·8182, or ·81818, thus coming nearer and nearer in value but never quite equivalent. The seventeenth century had travelled far from Boetian arithmetic, or a mind trained to appreciate the curious points which arise in numbers could surely have seen a repetition in these Hindu-Arabic figures. Addition, subtraction, multiplication, and division are worked as in whole numbers, the decimal point being omitted in the two latter and reintroduced in the answer by counting places. As the rules are given without any principle shown it takes five illustrations in multiplication and fourteen in division to attend to the different cases which arise. A rule for decimalizing money at sight is given.

There is a section on mensuration. The area of a rectangle being defined, the work proceeds through

the right-angled triangle and scalene triangle to the trapezium and polygon, with several diagrams to show how to cut up figures for measuring.

Cocker gives three readings for the ratio of circumference to diameter of a circle—*i.e.*, π. For working purposes 22 : 7, the value according to Archimedes, may be used, or 3·1416, if decimals are preferred. He also gives van Ceulen's value to twenty places of decimals. Cocker quotes correctly, but not from the latest source. Ludolf van Ceulen (Cologne, 1539–1610) had given the value of π to twenty places in 1596. Later he was able to carry the calculation to over thirty places in decimals, and this value was published in 1616, six years after his death. The number π is known in Germany as 'Ludolph's number.' Cocker's formula for the area of a circle is $\frac{1}{2}$ circumference $\times$ $\frac{1}{2}$ diameter.

Useful volumes are given. A brick is made up of 'dyes,' or little cubes, and measured accordingly. A cylinder is considered because it is the form of the rolling stone, the cone for a sugar loaf and round spire steeple, the frustum of cone or cylinder for tapering timber or brewers' conical tuns.

The work is interesting and good except for carelessness in not distinguishing between linear, square, and cubic feet. Examples in square roots are taken from such problems as finding the mean proportional of two numbers, the hypotenuse of a right-angled triangle, and the diameter of a circle from the area.

There is also a section on interest. Simple interest is taken as a problem in proportion, and compound interest as continued proportion. The tables of interest and annuities are given in decimals, and appear to have been copied as they stand from Wingate, omitting those for 4, 11, and 12 per cent.

CHAPTER VI

WARD AND MALCOLM

The Young Mathematician's Guide, by John Ward, 1707, is in five parts, arithmetic, algebra, geometry, conic sections, and arithmetic of infinites. The section on arithmetic occupies one hundred and forty pages of a large book, and therefore it is unlikely that it found its way to men who needed only arithmetic. There were several editions, the twelfth, published in 1771, being the book referred to in the following description.

In the introductory chapter the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 are called Arabick characters, but later they are said to be Hindu. A few signs are explained, among them $\div$, $=$, and $::$ (so is).

Ward sees the difficulties which beset the beginner, and goes fully into explanations and illustrations of the carried figures in addition and multiplication, of the exact value of all the partial products in multiplication, and of all the multiples of the divisor which appear in division. Of all the books examined this is the best since Wingate in clear exposition of the Hindu-Arabic notation in action. It is the first to try to explain subtraction. He gives initially the method by decomposition, and then, as an alternative, the customary method, known in recent years as the method of equal additions.

$$\begin{array}{r} 7496 \\ \underline{5849} \\ 1647 \end{array}$$

> Say 9 from 6 cannot be, but 9 from 16 and there remains 7 . . . then proceed to the place of the tens where you must now pay the 10 that was borrowed to make 6 16, by accounting the upper figure 9 in that place one less than it is, saying 4 from 8 and there remains 4.
>
> Or else (which is the most practised) say 1 I borrowed and 4 is 5 from 9, and there remains 4.

The word 'borrowed' dies hard. It has taken centuries to realize that in this second case two tens are brought in from outside, one as ten units to help out the 6 and the other as a ten to balance it in the lower line.

Another boon to the beginner is the multiplication table prepared for learning by heart. The Pythagorean table of multiples in either triangular or square form is economical for reference, but for memorizing the factors and products should be contiguous. Ward uses the multiplication sign and the sign for equality, and arranges the tables thus:

$3 \times 3 = 9$	$4 \times 4 = 16$	$5 \times 5 = 25$	$6 \times 6 = 36$	$7 \times 7 = 49$	$8 \times 8 = 64$
$3 \times 4 = 12$	$4 \times 5 = 20$	$5 \times 6 = 30$	$6 \times 7 = 42$	$7 \times 8 = 56$	$8 \times 9 = 72$
$3 \times 5 = 15$	$4 \times 6 = 24$	$5 \times 7 = 35$	$6 \times 8 = 48$	$7 \times 9 = 63$	$9 \times 9 = 81$
$3 \times 6 = 18$	$4 \times 7 = 28$	$5 \times 8 = 40$	$6 \times 9 = 54$		
$3 \times 7 = 21$	$4 \times 8 = 32$	$5 \times 9 = 45$			
$3 \times 8 = 24$	$4 \times 9 = 36$				
$3 \times 9 = 27$					

The example for practice in multiplication is

$$987654321 \times 123456789 = 121932631112635269.$$

In both multiplication and division involving large numbers Ward recommends making a table of the first nine multiples of the multiplicand or, in division, of the first nine multiples of the divisor. As the working of the problem proceeds the multiples, as they are required, are copied from the table. Tables of multiples were frequently used. Hylles, as we have noticed,

set them about the intersecting lines which he called a 'long sword.' Napier's rods, which are described later in this chapter, did away with the necessity for reckoning a table for every problem.

After the first four rules in abstract numbers there is a chapter on weight, measures, and money. Ward is nowhere so interesting as here. He is described on the title-page as Professor of Mathematics at Chester, but he was for a time Gauger-General to the Excise, and had a very full knowledge of weights and measures and their history. He refers to statutes which at different times defined them, and described experiments (which he himself had attended) made to keep the standard measures true and the experiment to try to find the length of one degree of the earth's surface. Later, in the chapter on alligation, in the problems of mixing metals he tells about the rules for inspecting our coinage to keep up its value, and gives a table of specific gravities of common substances to six places of decimals.

He does not find it easy to define time, so quotes from *Lucretius*, Book I:

> Time of itself is nothing, but from Thought
> Receives its Rise, by labouring Fancy wrought
> From things consider'd, whilst we think on some
> As present, some as past, or yet to come.
> No Thought can think on Time, that's still confest,
> But thinks on things in motion or at Rest.

The chapters on the fundamental rules in concrete numbers and on vulgar fractions do not go beyond Cocker except in style. Ward teaches the essentials fully, and, having given one good explanation, stops. Like Wingate, his clear illustrations almost of themselves suffice to teach the matter.

Ward mentions the decimal point and comma, but uses the comma. In multiplication the comma is

placed in the product by counting places. In contracted work the multiplier is reversed and two extra figures of the multiplicand retained for accuracy. There is more argument over division. First he refers to division in whole numbers to draw attention to the rule that "the Quotient figure is always of the same value or degree with that figure of the dividend under which the unit's place of its product stands." Another point is that the number of places of decimal parts in divisor and quotient together must always equal the number of those in the dividend. From these two rules there arise four cases:

(*a*) If the number of places in divisor and dividend be equal the quotient is a whole number.

(*b*) If the number in the dividend exceed those in the divisor, cut off the excess in the quotient.

(*c*) If the number in the dividend is less than in the divisor add ciphers to the dividend to bring up the number of places, and then the quotient will be a whole number.

(*d*) If, after division, there are not as many figures in the quotient as there should be, prefix ciphers to make up the number—*e.g.*,

957)7,25406(758 write ,00758

The rule for contracted division is:

> Having determined how many places of whole numbers there will be in the quotient, if any at all; or, if none, of what value or place the first figure of the quotient will be, then omit or dot off one figure of the figure of the divisor at each operation, viz: for every figure you place in the quotient, dot off one in the divisor having a due regard to the increase which would arise from the figure so omitted.

In reducing certain vulgar fractions—*i.e.*, thirds, sevenths, and ninths—to decimals he points out the

recurring figures in the quotients. He shows how to convert sums of money into decimals of a pound, and weights into decimals of a higher unit, and gives tables. He does not, however, teach the decimalization of money at sight.

Progressions are carried a little farther than in Cocker. Of arithmetical series he considers means and extremes, the *n*th term, and the sum; of geometrical series, the means and extremes, the ratio, and the sum. The sign ∺ means "continual proportionals" as for instance in "In any series of ∺ the ratio is found by dividing any of the consequents by its antecedent." From the time of Recorde and his costly horse, writers have always enjoyed a problem introducing geometrical series. Ward gives: "A cunning servant agreed with his master (unskilled in numbers) to work for 11 years with no other reward than 1 wheat corn in the first year growing in 10-fold proportion." In all he earns 111,111,111,110 corns. With wheat at 3 shillings a bushel and 7680 corns to the pint he earns £33,908 8*s*. 4½*d*. In the given solution, the payment for the first year is 10 corns, and the servant therefore had ten times his due payment.

"How to change or vary the order of things" is a chapter on permutations. The rule is stated thus:

> The method of finding the number of changes is by a continual multiplication of all the terms in a series of arithmetical progression whose first term and common difference is 1, and the last term, the number of things to be varied, viz.: $1 \times 2 \times 3 \times 4$ . . . .

He argues the truth of this from three or four things to be changed. The practical use is in bell-ringing. He gives as illustration the case of St Mary-le-Bow Church in Cheapside, which before the fire of 1666 had 12 bells, and calculates the time required to ring

all possible changes at the rate of two strokes a second.

Another example is: "6 gentlemen met at an Inn and were so pleased with their host that they said they would stay so long as they could every day sit in a different position at dinner." The host sits with them. The answer given is 13 years 291 days, which is correct if we reckon the 13 years to include four leap years. "A very pretty frolic indeed" is the comment. Ward omits the usual chapter on sports and pastimes.

In the Rule of Three and its applications Ward, like Cocker, puts the terms of the same kind, which he calls the *homogeneal* terms, in the first and third places; more decimals are introduced in the examples. The Rule of False Position is omitted because algebraic methods are better for such problems. Extraction of roots follows the ordinary course, though one example is worthy of mention. Find the cube root of

$$976379602989073960279630298890.$$

We have to look in the section on algebra to find simple and compound interest treated as examples in progressions. P, A, R, and t are the symbols, and Ward uses the power index for the higher powers—*e.g.*,

$$R \qquad RR \qquad RRR \qquad R^4 \qquad R^5.$$

The tables are reckoned for 6 per cent. interest, but, as a concession to a recent Act of Parliament which made 5 per cent. the maximum interest, he added some tables of his own calculation for that rate. He gives also a table of probabilities of life deduced, he says, from observations by Doctor Halley at Breslau and by Mr Smart in London.

This book pleases. It is outstandingly sounder in

teaching, better written, and more interesting than Cocker. It is a pity that the arithmetical part was not published by itself instead of being bound up with the higher branches of mathematics. As it was it had no opportunity of entering into competition with Cocker, and its direct influence on teaching was negligible.

Arithmetic Theoretical and Practical, by Alexander Malcolm, Teacher of the Mathematicks at Aberdeen was published in 1730. It is a large book of 643 pages and about 750 words to the page, and is a scholarly work with evidence on every page that Malcolm had read and thought and worked in numbers. He writes of them with no economy of words, presenting every view; and yet, allowing for the knowledge of the day when writers on arithmetic were still groping their way towards generalizations, there was little redundant matter. It is a book which stands alone, overwhelming in its argument and thoroughness.

Malcolm was more interested in abstract numbers than in their applications, and there is a comparative scantiness about the chapters on such matters as tare and tret, loss and gain, fellowship, barter, interest, and annuities. Also he demonstrated truths in general, that is in literal, terms to such an extent that on first opening the book looks like a work on algebra.

Malcolm is a critic, but a constructive critic. In the preface he names one or two recent works, and says where he thinks the report of them is overrated. Also of works in general he is not satisfied that there is sufficient demonstration of the rules, and he complains that teaching goes rather by rote than reason. He gives his reason for every departure from customary teaching. He rejects the Rule of False Position because

it is cumbrous, and because the solution of the questions usually solved by it is simpler by algebraical methods. In the choice of terms he rejects the word *concrete* in favour of *applicate* numbers, because his word is self-explanatory. He teaches the simple rules in applicate numbers after, and not simultaneously with, the first four rules in abstract numbers because even in the first rule of addition, in money for instance, there is need to divide by 12 and 20. Throughout the book he analyses every rule and theorem, arguing and presenting it in all lights, choosing the best way of teaching it and giving reasons for his choice. To-day such a book would be in a training college or a teacher's library.

In the first chapter Malcolm refers to the idea still current at his time that numbers began with 2. "One's none, two's some" begins the old jingle of the numbers, and men of the Middle Ages could solve the following riddle:

> I came to a tree where were apples.
> I ate no apples,
> I gave away no apples,
> Nor I left no apples behind me:
> And yet I ate, gave away, and left behind me.

Malcolm says there has been a great deal of argument filled with abundant idle and nonsensical jargon about this. He gives the reasons which have been put forward from time to time for and against unity and multitude being distinguishable. Euclid, he says, who defined number as a multitude of units, always treated of unity under the same name; and it would be impossible to write the science of numbers, or arithmetic, without mentioning unity. He himself, therefore, prefers to put *one* with the numbers.

Notation and the first four rules occupy nearly

sixty pages. Tables are given for the addition, subtraction, and multiplication of units.

Addition Table

1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	10	1
	4	5	6	7	8	9	10	11	2
		6	7	8	9	10	11	12	3
			8	9	10	11	12	13	4
				10	11	12	13	14	5
					12	13	14	15	6
						14	15	16	7
							16	17	8
								18	9

The numbers to be added will be found, one at the head of a column, the other at the right-hand end of a row. The sum is the number where the column and the row intersect.

Subtraction Table

1	2	3	4	5	6	7	8	9	0
0	1	2	3	4	5	6	7	8	1
	0	1	2	3	4	5	6	7	2
		0	1	2	3	4	5	6	3
			0	1	2	3	4	5	4
				0	1	2	3	4	5
					0	1	2	3	6
						0	1	2	7
							0	1	8
								0	9

The minuend, or number which is to be diminished, will be found at the head of a column, and the subtrahend, or number which is to be subtracted, at the right-hand end of a row. The difference is the number where the column and row intersect.

Multiplication Table

1	2	3	4	5	6	7	8	9	1
	4	6	8	10	12	14	16	18	2
		9	12	15	18	21	24	27	3
			16	20	24	28	32	36	4
				25	30	35	40	45	5
					36	42	48	54	6
						49	56	63	7
							64	72	8
								81	9

This table is a modification of the table of Pythagoras.

At last two methods of subtraction are clearly written.

(*a*) If the figure in any place of the minuend is less than its correspondent in the subtractor, *add ten* to that figure and subtract from the sum and set down the digit which remains: then *add one to the next* figure of the subtractor . . . .

(*b*) When taking 26 from 52, 52 can be treated as 40 + 12, and the 6 taken from the 12.

Several points are discussed under multiplication, namely,

$A \times B$ is the same as $B \times A$, A is $1/B$ of AB, factors are aliquot parts. Also if $A = B + C$, then $AD = BD + CD$, and if in turn $D = E + F$ then $AD = BE + BF + CE + CF$.

This is also one of the very few arithmetical books giving a description of Napier's rods. This is merely a mechanical device for showing products; Napier's greater service to multiplication was the invention of logarithms.

Napier's rods were eleven in number, the index rod and the ten rods giving multiples of all the numbers.

To multiply 4768 by 385, lay side by side the index rod and the rods headed 4, 7, 6 and 8 (see p. 99).

1	1	2	3	4	5	6	7	8	9	0
2	/2	/4	/6	/8	1/0	1/2	1/4	1/6	1/8	/0
3	/3	/6	/9	1/2	1/5	1/8	2/1	2/4	2/7	/0
4	/4	/8	1/2	1/6	2/0	2/4	2/8	3/2	3/6	/0
5	/5	1/0	1/5	2/0	2/5	3/0	3/5	4/0	4/5	/0
6	/6	1/2	1/8	2/4	3/0	3/6	4/2	4/8	5/4	/0
7	/7	1/4	2/1	2/8	3/5	4/2	4/9	5/6	6/3	/0
8	/8	1/6	2/4	3/2	4/0	4/8	5/6	6/4	7/2	/0
9	/9	1/8	2/7	3/6	4/5	5/4	6/3	7/2	8/1	/0

Beginning multiplication from the right-hand end.

Against index 5 is	23840
Against index 8 is	38144
Against index 3 is	14304
Therefore the product is	1835680

Long division is used. There is also division by factors, not very economically arranged—*e.g.*, 479 ÷ 60.

3)479(159
rem 2

5)159(31
rem 4

4) 31(7
rem 3

7 times with remainder 59.

1	4	7	6	8
2	/8	1/4	1/2	1/6
3	1/2	2/1	1/8	2/4
4	1/6	2/8	2/4	3/2
5	2/0	3/5	3/0	4/0
6	2/4	4/2	3/6	4/8
7	2/8	4/9	4/2	5/6
8	3/2	5/6	4/8	6/4
9	3/6	6/3	5/4	7/2

A much closer approximation to the present arrangement of work is seen in finding the prime factors of a number.

2	68796
2	34398
3	17199
3	5733
3	1911
7	637
7	91
	13

The usual tables of money, weights, and measures and the rules in applicate numbers are given. Units, that is, denominations, must be respected.

> Suppose then that it is proposed to multiply 8 l. by 3 l. I should ask the proposer, what he means by multiplying 8 l.? He can make no other answer but that it is the repeating of 8 l. a certain number of times. . . . Then I ask him, How many times he would have it repeated? And if he answers 3 l. I hope the absurdity is manifest.

The general theory of fractions is explained in algebraical symbols before the rules are given. The rules follow the ordinary course, except in the arrangement of division.

$$\frac{3}{39}\Bigg)\frac{6}{13}\Bigg(\frac{234}{39}=6$$

The point (·) is used in decimals, and the rules on the whole are as in Ward. There are also the methods of bringing one applicate number to the decimal of another, followed by tables of decimals of pounds (money), grains, pints, days, etc. The rule for decimalizing money at sight is given. Later there are cases of decimals which do not terminate. Some are *certain*, that is, have repeating figures, others are *uncertain* going on without a repetition of figures. Some he evaluates in vulgar fractions, and so is able to multiply and divide them.

The treatment of powers and roots, like that of vulgar fractions, is too difficult for any but students. It is in chapters like these that Malcolm fails in having a two-fold intention in the book. The student could enjoy all he wrote, but the men of business mentioned in the title as other probable readers would leave considerably over half unread if indeed they were able to select what to read and what to omit. The extraction of roots is based on the algebraical powers of a binomial, and indices are used throughout the chapter —*e.g.*, A^4 instead of $A\,.\,A\,.\,A\,.\,A$ and $A^{\frac{1}{3}}$ for $\sqrt[3]{A}$.

The book on proportion, that is, on progressions, is more advanced than in any arithmetic book up to this time. He treats of arithmetical, geometrical, and harmonical series. Some new facts about series are given, among them that the cipher (0) may be the first term of an arithmetical series, an arithmetical progression can be raised increasing *ad infinitum* from any given number but not decreasing because by continual lessening of any number you may at last exhaust the whole, and unity (1) is the geometric mean between a number and its reciprocal.

A number of theorems on arithmetical series are built up, and later when l and d are introduced for nth term and difference respectively our essential formulæ $l = a + \overline{n - 1} \,.\, d$ and $S = \frac{(a + l)n}{2}$ emerge. The day had not arrived when these sufficed, and case on case is piled up until the reader is left with directions for solving each of the following problems:

	1	2	3	4	5	6	7
GIVEN	a, l, n	a, l, d	a or l, n, d	a, l, S	a or l, n, S	d, n, S	a or l, d, S
SOUGHT	d, S	n, S	l or a, S	n, d	l or a, d	a, l	l or a, n

The connexion $a + d = b + c$ holds between the terms a, b, c, and d of an arithmetical series.

Similarly in geometrical progression the following problems are solved:

	1	2	3	4	5	6	7
GIVEN	a, l, r	a, l, n	a, l, S	a or l, r, n	S, n, r	a or l, S, r	a or l, n, S
SOUGHT	S, n	S, r	n, r	l or a, S	a, l	l or a, n	l or a, r

There is a section on the properties of numbers. It deals with prime and composite numbers, commensurate and incommensurate, odd and even, perfect

and imperfect, deficient and abundant, and the figurate numbers which build up shapes.

Commercial arithmetic is treated very simply and no fuller than in any other book. Malcolm advocates understanding the arithmetical rules thoroughly, especially that of proportion, and then examining the sense and import of each question. Having said that there is little else left for him but to go through fellowship, factorage, loss and gain, bartering, tare and tret, alligation, and exchange as others had done before him. The rules of simple interest are obtained by Rule of Three and then given, not fully in words, but by formulæ in the obvious letters A, p, r, and t. Compound interest is worked by the formula $A = pR^t$ ($R = £1{\cdot}06$, the amount per pound at the close of the year). There is a discussion of discount and annuities, but not the many tables given in Wingate and others.

William Webster was a writer on arithmetic and book-keeping in the first half of the eighteenth century. The book described here has on its title-page:

> *Arithmetick in Epitome* or a compendium of all its Rules both Vulgar and Decimal. In two parts. Intended for the Remembrance and further Improvement of Those who have already made some Progress in this usefull science.

It has also the advertisement of the school 'By William Webster, writing Master and Accomptant, at the corner of Cecil Court, on the pavement in St Martin's Lane, Where Youth may Board.' Part I is undated, Part II has a separate title-page and is dated 1715. The index of both parts is at the end of the book.

The title-page of Part I and parts of other pages are engraved. It is done, Webster says, for the sake of

accuracy, the letter-press being uncertain, and no care has been wanting in its correction nor charge spared to render the book complete. Part II abounds in tables. There are the decimal tables of coin, weight, measure, and time, the table of days showing the number between any day of any month and the same day in another month, and tables of compound interest, annuities, and present worth. They are all very finely engraved by one Bickham, and must have added considerably to the cost of the book.

The foreword is addressed to clerks and accomptants, and the book is of the nature of a business arithmetic, with special attention to tables, practice, and brief ways of working. It is, as the title-page points out, a summary for revision or for reference, but there are in it, for those who need them, references to writers who have given fuller explanation and teaching of the subject. It is a very useful book for its purpose and has only one serious fault: the statements in the Rule of Three do not make sense.

Six methods of division are shown: three old methods differing in the position of the divisor, and three Italian methods where partial remainders may be set above or below the dividend, or partial products may be omitted. There is also short division.

Webster's teaching follows the current methods, and only one or two curious points need be mentioned. Cancelling is allowed to simplify reckoning in problems of Rule of Three, but in reduction of fractions, where the greatest common measure is not found by dividing out, the work goes down in steps.

$$\frac{192}{576} \quad \frac{96}{288} \quad \frac{48}{144} \quad \frac{24}{72} \quad \frac{12}{36} \quad \frac{6}{18} \quad \frac{3}{9}$$

Cloff, or clough, the allowance for retail selling, is

said in so many words to be the allowance 'for the turn of the scale.'

An example in discount is:

> A Bill of Exchange for 250 l. is dated at Amsterdam, June the 13th New Stile: at Usance is accepted, and payment offer'd the 15th ditto Old Stile, what must be then receiv'd, Rebate being made at 6·1 per cent. per annum?
>
> *Note.* New Stile is 11 Days before Old Stile.

CHAPTER VII

VYSE AND CHAPPELL

In the middle of the eighteenth century Cocker was still the popular arithmetic, but, whether in reaction to his pomposity or not, there were attempts to enliven the subject by throwing it into verse. The first of the books, *Arithmetic*, by Solomon Lowe, 1749, is far from satisfactory, its great fault being that the rules are given in alphabetical order. The verse is forced, but Hylles has already warned us that the mathematical sciences do not lend themselves to poetry. Lowe's rule for Barter is this:

> What's to be changed, Value; than see what
> That will purchase of t'other:
> If an advanced price of one, a proportionable
> Find for the other.

A much better book was *The Tutor's Guide*, by Charles Vyse, which came out about 1771 and ran through several editions. This, though not the first, was one of the earliest books written specially for the teacher. *The Tutor's Guide*, *The Schoolmaster's Assistant*, and *The Tutor's Assistant*, by Charles Vyse, Thomas Dilworth, and Francis Walkingame respectively, together with Cocker and the new Bonnycastle were the standby of the country schoolmaster in the early years of the nineteenth century. Vyse's teaching is good, and his verses ingenious. He quotes, however, from others without acknowledgment, and what should have ruled out his books from any school, namely, coarse humour, mars all his work. Three problems suffice to illustrate his verse.

When first the marriage-knot was tied
Between my wife and me,
My age to her's we found agreed
As three times three to three;
But when ten years, and half ten years,
We man and wife had been,
Her age came up as near to mine
As eight is to sixteen.
Now, tell me, I pray,
What were our ages on the wedding day?
[Not original.]

Once as I walked upon the banks of the Rye
I, in the Meads, three beauteous nymphs did spy,
Saying "Well met, we've business to impart
Which we cannot decide without your Art:
Our Grannum's dead, and left a Legacy,
Which is to be divided amongst three:
In Pounds it is two hundred twenty-nine,
Also a good mark, being sterling coin."
Then spake the eldest of the lovely three,
"I'll tell you how it must divided be;
Likewise our names I unto you will tell,
Mine is Moll, the others Anne and Nell.
As oft as I five and five-ninths do take,
Anne takes four and three-sevenths her Part to make:
As oft as Anne four and one-ninth does tell,
Three and two-three must be took up by Nell.

A castle wall there was, whose height was found
To be an hundred feet from th' Top to th' ground:
Against the wall a ladder stood upright,
Of the same length the castle was in height.
A waggish youth did the ladder slide,
(The bottom of it) ten feet from the side;
Now I would know how far the top did fall,
By pulling out the ladder from the wall.

Richard Chappell in *The Universal Arithmetic*, 1798, tried to encourage the practice of subtracting in long division without writing the subtrahend. This method had been taught by William Webster in 1715, by George Fisher in 1744, who called it the short Italian method, and by Solomon Lowe.

The verse runs a little more smoothly in Chappell, particularly in the long tables. The multiplication table of fifty-six lines begins:

Three 3's is 9, three 4's is 12,
Three 5's is 15 sure;
And 3 times 6 is just 18,
And wants 2 of a score,

and contains the historical reference:

So 5 times 8 were 40 Scots,
Who came from Aberdeen;
And 5 times 9 were 45
Which gave them all the spleen.

The pence table is

Twenty pence is one and eight pence,
This in paper out I laid;
Thirty pence is two and six pence,
This I for a grammar paid.
Forty pence is three and four pence,
This I spend in going to School;
Fifty pence is four and two pence,
And with this I bought a rule.
Sixty pence is just five shillings,
Master this for entrance got;
Seventy pence is five and ten pence,
Who dare say that it is not?
Eighty pence is six and eight pence,
This is just a lawyer's fee;
Ninety pence is seven and six pence,
As you all may plainly see.
A hundred pence is eight and four pence,
This I lent to cousin Ben;
But as he wanted nine and two pence,
I gave him the other ten.
Yet had he wanted just ten shillings,
Then I must lend him ten more;
I bid him take it and not plague me,
But never borrow as before.

Among the problems are:

Two ships upon the equinoctial line
One rang the bell at six,—th'other at nine.
What's distance in English miles pray tell to me
And for so doing you shall rewarded be.

As I was trailing over some pasture grounds,
Up starts a Hare before my Beagle hounds;
The dogs being quick on scent, did swiftly run,
Unto her nineteen rods, just twenty-one;
The distance that she started up before
Was measured ninety rods, no less nor more;
Now unto me I'd have you this declare
How far they ran before they caught the Hare?

'Twas on a clear day, the sun shining bright,
I saw a long May-pole standing upright.
To know the length, I a measurement made,
By measuring a Stick and the length of the shade;
A staff of five feet, made three feet in length,
The shade of the pole was forty feet one-tenth;
The length of the pole I beg you will show,
For that's the answer I desire to know.

Ingenious youths I pray engage
Your thoughts sometime to find my age;
From what's below I make no doubt,
But you may easy find it out.

Multiply half my age by half a score,
And from that product take seventy-four;
Add to it four, divide by three,
What is my age, you'll plainly see.

These books are an honest attempt to write in simple words and to interest school pupils. But for this it would be difficult to regard them as serious arithmetic. They are too often flippant and not in good taste. They were written when arithmetic was at its lowest repute. Though a somewhat wider range of problems was beginning to come into the subject, it had had a commercial bias since the days of John Mellis, and was now so overburdened with these topics as to be out of odour with many.

Life has other hopes than Cocker's
Other joys than tare and tret.

Also the rules had always been taught by rote seldom justified by reason, and Cocker had exagger-

ated the empiricism. He and his followers had overlaid the rules with a mass of words; and lack of thought in statement, contradictory repetition, and careless naming of units were found too frequently. Could there be dignity enough left in the subject to preserve it from third-rate verse?

CHAPTER VIII

BONNYCASTLE, DE MORGAN, SANG, AND OTHERS

THE arithmetic by John Bonnycastle, first published in 1780, was called *The Scholar's Guide to Arithmetic or a Complete Exercise Book for the use of Schools.* It became a popular work, being published in England several times over a period of fifty years. It was also reprinted in America as early as 1786. By this time the books were becoming more definitely associated with schools and school teaching, and Bonnycastle's presents a distinct step forward in the teaching of arithmetic. The book available for description is the larger work, *An Introduction to Arithmetic or a Complete Exercise Book for the use both of Teachers and Students.* By 1810, when this book was published, Bonnycastle was Professor of Mathematics at the Royal Military Academy, Woolwich.

The preface is a brief history of arithmetic up to his own day, when Bonnycastle thought it was "one of the few sciences so extended and improved that little more remains to be done in it except by giving its rules and processes a more commodious and accurate form." He instances particularly the Rule of Three as needing attention, "the terms being so disposed that a comparison is made between unlike quantities." Nearly always these statements had been unintelligible, and only the accident of the product of two terms being the same if they are interchanged saved the answer. Bonnycastle's statements whether in Direct or Indirect Rule of Three are all intelligible.

There is a table of contents of eighty-four items in which the following new topics occur: bills of parcels, circulating decimals, duodecimals, stocks, purchasing freehold estates, permutations and combinations, and a table of prime numbers.

The work proceeds by definition, rule, illustration, proof, examples for practice. It was still thought best to present it in this logical and orderly way, but only a little later De Morgan said that it was better teaching to begin by suggesting a problem and arguing the rule from it. The text is simple and clear, and at the foot of the pages there are copious footnotes, often giving the general case in symbols. This arrangement allows the reader to take all the plain arithmetic on his first reading and come back over the footnotes when familiar with the work and also with algebra, or alternatively it parts the pupils' portion from the teacher's. The faults of the book were those of the day: exercises involving large numbers and over-meticulous accuracy—for example, the distance between two places is worked out to be 26 miles 2 furlongs 24 poles 2 feet and 1 inch.

The directions on the subtraction difficulty are clear, and a good illustration is given in both abstract numbers and compound subtraction. Topics other than commercial are drawn upon for the examples. For example:

> Homer was born 2520 years ago, and Christ 1809. The years between?
>
> The mariner's compass was invented in 1302, printing in 1440, and America was discovered in 1492. The years between?
>
> Gunpowder was invented in 1344, and the Reformation commenced in 1517. The years between?

The multiplication table is carried to 12, and multi-

plication and division by factors and also shortened division, by omission of the multiples of the divisor, are taught.

The chapter on compound division is weak. The definition is "Compound Division is the method of finding how often one given number is contained in another of different denominations." This is a lapse not expected from one so insistent on correctness of wording elsewhere. What he does later is, in fact, to find the *n*th part of a sum of money. The other case of division, to find how often one quantity is contained in a homogeneous quantity, comes in among the problems following reduction. Among these problems we find, "How long would it require to count five hundred millions of money, which is the National Debt of this country at present, at the rate of 100 l. a minute without intermission?" The Rule of Three, practice, tare and tret, and bills of parcels follow immediately upon these fundamental rules in concrete quantities.

There is nothing new in Bonnycastle's treatment of vulgar and decimal fractions, except that the divisor in vulgar fractions is inverted. In connexion with the reductions of fractions the tests for divisibility by 2, 4, 8, 3, 6, 9, 5, 10, and 11 are given, together with the fact that if a number cannot be divided by a number less than its square root it is prime. Decimalization of money at sight is taught with the corrections for more than 12 and 37 farthings. All work in circulating decimals is done by conversion into vulgar fractions.

Duodecimals or cross-multiplications are the ready reckonings of artificers. The dimensions are given in feet, inches, and twelfths of an inch, so that twelve in one column counts as one to the next. The result is stated in the same, *i.e.*, linear, measure, and approximates to the correct result if read in square measure.

For example, multiply 4 feet 7 inches by 6 feet 4 inches.

	4 feet	7 inches	
	6 feet	4 inches	
Multiply by 6 feet	27	6	
Multiply by 4 inches, setting result one place to the right	1	6	4
	29 feet	$0\frac{1}{3}$ inches	

The order of the topics becomes confused from this point. Powers and extractions of roots are taken in the usual way, deduced from the algebraical expression. Then follow stocks, a new subject in elementary arithmetic books. Stocks, Bonnycastle teaches, are of different kinds, such as bank stock, 3 per cent. reduced, 3 per cent. Consols, omnium, etc., the prices of which vary according to the circumstances of the times and the rumours that prevail with respect to war and peace. When the price of 3 per cent. Consols is said to be $63\frac{1}{8}$, the meaning is that £63 2*s*. 6*d*. must be paid on that day for £100 of this stock: and in this case the purchaser may be considered as having £100 in the bank for which he is to receive £3, or two half-yearly dividends of £1 10*s*. each; and so for any larger sum. Omnium is the term that denotes the several kinds of stock, lottery tickets, etc., by which Government pay those who agree to advance a certain sum of money, by way of a loan; and when it is said to be at so much premium, as for instance $1\frac{1}{2}$ per cent. the meaning is that if a person purchases £100 of this loan he must pay £1 10*s*. more than the original lender. No examples are given on any but Government and East India Stock, and there is no mention of any similar business on the part of trading companies.

The next subjects in order are: simple interest and discount, compound interest built up year by year by

the addition of the simple interest of the year, equation of payments, barter, profit and loss, fellowship and alligation, and then a return to interest. This time decimals and formulæ are used, with a recommendation of logarithms in the footnotes. He touches upon freeholds and annuities respited and in reversion. The tables often found in books since Wingate's are not included here.

Progressions are treated in the modern way, stated fully in words in the text and given by the corresponding formulæ in the notes. Strangely enough after so much knowledge of algebra has been presumed, the old Rule of False Position is given and with a wider range of problems than hitherto.

Permutations and combinations are applied to many interesting matters, such as bell-ringing, sitting at table, and shuffling cards. Bonnycastle's problems are of infinite variety and always worth examining.

> 1. How many changes can be made of the words in the following verse: Tot tibi sunt dotes, virgo, quot sidera caelo: without spoiling the sense or meaning of the verse?
>
> Ans.: 3312 changes.
>
> 2. How many permutations and combinations in the form of words can be made of the 26 letters of the alphabet, taking them by two's, three's, etc., up to 26?
>
> Ans.: 64,098,086,171,687,043,724,249,756,415,443,927.

The French metric money is mentioned and also that of America.

> In North America and the West Indies accounts are kept in pounds, shillings, and pence as in England. But as there are here but few coins, they are obliged to substitute a paper currency for carrying on their trade: which, being subject to many casualties, suffers a great discount in its negociation.

The last seventy-five pages on properties of numbers, systems of notations, recreations, tables of squares, cubes, and roots, and of prime numbers must have been an interest in many a leisure hour. It is one of the best examples of modern theoretical arithmetic. Experience, as well as Euclid, Boetius, and others, contributed to the theorems about odd and even, prime, perfect, square, and cube numbers. One example of his theorems is:

> *Every cube number is of one of the forms $7n$ or $7n \pm 1$.* For any number being divided by 7, must leave for a remainder either 0, 1, 2, 3, 4, 5 or 6. That is, every number is of one of the forms $7n$, $7n + 1$, $7n + 2$, $7n + 3$, $7n + 4$, $7n + 5$, $7n + 6$. And if each of these forms be cubed they would evidently leave the same remainders when divided by 7 as the cubes 0^3, 1^3, 2^3, 4^3, 5^3, 6^3. But these last being divided by 7 have for remainders 0, $+1$ or -1; and therefore every cube number is of one of the forms $7n$ or $7n \pm 1$. Corollary. Hence if any number, when divided by 7, leaves a remainder of 2, 3, 4 or 5 that number is not a cube.

The greatest prime number yet (1810) found is 2,147,483,647, discovered by Leonhard Euler. At the end of the book there is a list of all the prime numbers in the first 100,000 numbers. Many of us have put down the primes among the early numbers to find that there are 26 in the first hundred, 21 between 100 and 200, and 16 between 200 and 300, and have the idea that the primes get gradually fewer in the higher ranges. This is not the case. To take a few cases, between 69,500 and 69,600, and also between 79,700 and 79,800 there are only three prime numbers, while in the ranges of much higher numbers 95,200–95,300, 95,700–95,800, and 96,700–96,800 respectively there are twelve primes, and in the ranges 83,200–83,300 and 90,000–91,000 there are thirteen.

The Elements of Arithmetic by Augustus de Morgan was first published in 1830. Later editions had fuller chapters on square root, proportion, permutations and combinations, decimalizing money, scales of notation, and verification of work.

The plan in all editions was to divide the book into two parts; a full first part on the principles of arithmetic or the rules in abstract numbers, and a short second part, called commercial arithmetic, beginning with the tables of money, weights, and measures and going on through Rule of Three and practice to the applications of arithmetic.

The book was intended for schoolmaster and pupil, to bring more intelligible methods into the classroom. De Morgan was aware of the faults in current teaching and also the difficulties of the schoolmaster. It was his sympathy with the teacher on the problem of finding sufficient exercises for his pupils and correcting them which led him to write in his preface:

> I have also added 6 or 7 examples to each rule accompanied by the answer. These will be enough for any single pupil, but may not be considered sufficient for a School. To obviate this objection, I proceed to collect some expeditious modes of forming questions of which the answers should be readily known.

Then follow suggestions: varying the question without affecting the answer for half the class, and making use of the many tables printed of squares, cubes, roots, interest, and annuities. He also recommends keeping a book in which the first boy to solve a problem may copy in his solution and sign his name. "Besides the emulation thereby excited, a collection of examples would be obtained for future use."

A few years later in *Arithmetical Books* De Morgan describes a book brought out in 1830, the year of his

own *Arithmetic*, which has an ingenious way of helping the teacher. Even its title suggests a time-saving in teaching: *An Elucidation of the Tutor's Expeditious Assistant*, by Joh. White. The idea is that the figures in the answers of the problems should have some definite relation to one another that catches the teacher's eye; for example, in multiplication the figures descend by twos, as 86420, 20864, or 75319. The writer proposes this in all seriousness, and claims it to be a new discovery in arithmetic. One wonders what was its effect on the pupils; whether they learnt to check themselves by it in lessons, and whether there came to them in after years an anxiety if a product did not ring true.

In De Morgan's day in many primary schools arithmetic was taught to a class orally, with the examples for practice, revision, or testing delivered in the same way. This continued until the close of the century. Fifty years ago, however, it was possible to get packets of printed cards of exercises on special rules or selections from several rules, each packet containing one answer card. These were handed out and collected during the lesson. Later the handbooks for teachers greatly increased the number of exercises, and put the answers to all at the end of the book. Later still these exercises were published separately for pupils, the teacher holding the larger book with the notes on teaching and the answers. Thus the difficulty of setting sufficient exercises and correcting them has been amply overcome.

The books published before De Morgan's time introduced each topic with definition and rule. This may be defensible when writing for the older student who comes to the work with some inkling of its purport, but for a class of beginners it is time wasted. De

Morgan recommends class teaching, the lesson beginning with an explanation by the master of one or two simple examples, and

> when the principle has been thus discussed, let the rule be distinctly stated by the master, or by some of the more intelligent of the pupils, and let some very simple examples be worked at length. The pupils may then be dismissed to try the more complicated exercises with which the work will furnish them.

The foregoing two points show us De Morgan as friend and counsellor first of the teacher and then of the child, and we can safely say that he brought an ease into the classroom that was not there before.

Perhaps De Morgan insisted more than any other writer on a thorough knowledge of principles. Also in his own illustrations the statements are always precise, the denominations of concrete numbers are always respected, and there is always clear thinking. His contribution to arithmetic, beyond the historical work for which he is noted, was rather in improved methods of teaching than in additions to the rules. His aims were right, and his suggestions valuable. Yet this book, written specially for teachers and pupils, is more difficult than the treatise for adult readers by Lardner, his contemporary, because the latter book was free from mathematical symbols and written in simple language to the level of ordinary minds.

The part of De Morgan's teaching which has not been adopted is the introduction of the general case and algebra in every topic. Even in the fundamental rules, in numeration, there are discussions of the truths:

$$(a + b) + (a - b) = 2a,$$
$$(a + b) - (a - b) = 2b,$$

and

$$\frac{a^2 - 1}{a - 1} = a + 1.$$

The beginnings of arithmetic are now taught to mere children, and it is sufficient burden to a child to learn three notations—the printed letter, script, and the Hindu-Arabic—without adding a fourth by giving the letters a mathematical sense. Teachers may think also that a child is spending his time sensibly in collecting and grouping individual experiences, and it is unwise to force generalizations on immature experience.

D. Lardner's *Treatise on Arithmetic* was not intended to be a school book. It was written for the Natural Philosophy section of the *Cabinet Cyclopædia*, and appeared in 1834. Yet studies such as this and Malcolm's of a century earlier do influence teaching, even if only indirectly.

The treatise begins with the earliest notions of numbering. Then, coming to the Hindu-Arabic notation, it explains all the early rules with the use of counters in almost sufficient detail to satisfy modern kindergarten requirements. In particular there is the most complete explanation of subtraction, both by decomposition and by equal additions carried out with counters.

Coming to decimals, in multiplication the rule of placing the point by counting is compared with the process in vulgar fractions, where after multiplying together the numerators the result is placed over the product of the denominators. Lardner gives for division of decimals yet another suggestion, which is argued in this way: (1) the integral part of the quotient comes when the decimal places of divisor and dividend are equalized by the addition of ciphers to the shorter of them; (2) for the fractional part add sufficient ciphers (n ciphers—*i.e.*, multiplication by 10^n) to the dividend to bring out the division. Then

as the quotient will be 10^n times too much, mark off n places by the decimal point.

In the tables of complex numbers all the French metric measures adopted after the Revolution are given.

All the rules in whole and fractional numbers and in abstract and concrete, or complex, as Lardner calls them, are treated fully with great regard to the understanding of the principles. After this he is able, as was De Morgan, to shorten the part on applied arithmetic. Instead of providing a separate rule for every problem arising in interest, discount, profit and loss, brokerage, commission, insurance, tare and tret, and partnership, he says that the worker "must, from the conditions given, collect what are the arithmetical operations which should be performed on the numbers proposed to bring out the solution." Each exercise is therefore analysed. For example: "A merchant presents a bill of exchange for 3000 l. payable at the end of one year to his banker to be discounted."

Analysis. It is evident that the question here to be determined is, what is that sum of money which being now placed at interest would at the end of one year be worth £3000? for such is the sum which the banker ought to pay to the merchant; and the difference between this sum, whatever it be, and £3000 is the amount of the discount. The rate of interest is 6 per cent. A present sum of £100 would at this rate increase to the amount of £106 at the end of one year; the question therefore is if £100 at the end of one year become £106, what sum at the end of one year would become £3000? The statement would be thus

$$£106 : £3000 = £100 : x.$$

N.B. Lardner is employing true discount in this problem.

The Rule of Three as generally taught comes in for criticism. A ratio can exist, says Lardner, only between two quantities of the same kind, and though in using the quantities as abstract numbers an interchange of terms produces a right result, we divest the statement of all propriety and distinct meaning. Thus Bonnycastle, De Morgan, and Lardner, all in the early nineteenth century, were making a stand for clear thinking and right expression.

Edward Sang's *Elementary Arithmetic*, 1856, is interesting and refreshing. Sang must have loved numbers, and he was evidently an expert calculator. The majority of his readers, however, would not have concentration enough for such computing as he now and again recommends. Then, again, our attention is held by several novelties in treatment, and though an impression gains ground and gradually becomes a conviction as we read on, it comes as a shock to find it put into words on page 115, "Indeed we never have recourse to the general method when we can avoid it." To be appreciated the book should be read with a knowledge of the limitations of those before it rather than of the extensions of those since. For instance, we know well nowadays that counting and the elementary rules can be taught with material things, beans, nuts, acorns, and the abacus. Again, where Sang could teach that, as a merchant has goods coming in and going out of his warehouse all day, the details can be entered as they occur and all be collected at the end, we can use a plus (+) and minus (—) sign, and add and subtract the terms unsorted and as they happen to fall. We can retain the point (·) in multiplication and division of decimals, and keep the digits of the multiplier in their original order in contracted

work. But to see these and numerous other changes all made in one small book marks that book as refreshing.

We have not altogether adopted Sang's suggestion that multiplication should begin at the left-hand end of the multiplicand, which method required a few pages to explain how far to look ahead for the carrying figures.

There are many strange connexions shown in numbers. Incidentally in the chapter on multiplication he builds up the multiples of 7, pushing each one two places to the right, and finds on addition the set of figures which recur in the decimal equivalent of $\frac{1}{7}$.

$$
\begin{array}{l}
14 \\
28 \\
56 \\
112 \\
224 \\
448 \\
\hline
142857142848
\end{array}
$$

Sang's *Higher Arithmetic* includes chapters on indices and logarithms. John Napier (1550–1617) gave much thought to the simplification of the work entailed in multiplying and dividing long numbers. By 1614 his work was published explaining the nature of logarithms by a comparison between corresponding terms of an arithmetical and a geometrical series. This work was more concerned with the nature and possibilities of logarithms than with choosing the all-round most convenient base to which to compute. After the book came out Briggs had the idea that as the basis of our notation is decimal, there would be advantages in using logarithms computed with 10 as base. He wrote to and then went to confer with Napier, to whom perhaps some inkling of this improvement had already

come. Napier suggested that the logarithm of 1 should be 0 and that of 10 1. Briggs approved and, as he tells us, "rejecting what I had prepared, I set to calculate these." In 1624 his *Arithmetica Logarithmica* was published containing the table of common logarithms of 1 to 20,000 and of 90,000 to 100,000 to fourteen places. So far the idea and the calculation of logarithms was English. Then in 1627 Vlacq at Gouda filled in the gap left by Briggs and published a table of logarithms of the first 100,000 numbers to ten decimal places, which is carrying them far enough for the ordinary worker. In 1742 James Dodson published his converse of the ordinary tables, the antilogarithms. Then in the nineteenth century, about 1871, Sang designed and arranged tables of both logarithms and antilogarithms. His work for logarithms inspired his fuller treatment of indices. The mechanical devices of the slide-rule and ring were invented early in the seventeenth century, very soon after the use of logarithms became known. Logarithms are a late entry in arithmetic books, and only recently have been included in the ordinary school syllabus.

The four books described, those of Bonnycastle, Lardner, De Morgan, and Sang, all made distinct contributions to the teaching of arithmetic, though not all of them could claim popularity. There was another type of book peculiar to the latter half of last century, and found in most school subjects. It was filled with lists of bare facts or outlines of the subject. *The Manual of Arithmetic* by J. A. Galbraith and S. Haughton, 1858, is such a book. It is good only if good oral teaching accompanies it. It is in small print on well-arranged pages. Regardless of De Morgan's suggestions, it goes, with a great economy of words, steadily through definition, rule, illustration,

and examples for practice, and leaves the impression, by judicious omission, that the subject has no difficulties. There is no advance in teaching over Bonnycastle, but one new rule is included. Certain problems depend on more than one connexion, and these are solved by what is known as the Chain Rule. The section on exchange is shortened. There is a note that Canada had adopted the monetary system of the United States, and her accounts were kept in dollars and cents.

The Theory of Arithmetic, by David Munn, 1871, is intended as a supplement to the ordinary text-books, and is useful to teachers because it aims at giving a clear understanding of the principles underlying the various rules. It contains a good chapter on approximations and relative errors.

The Scholar's Arithmetic, by Lewis Hensley, 1873, was also written for pupils who had had a preliminary course. In the preface Hensley wrote, as did others of that time, that his aim was to promote clearness of conceptions and firm grasp of principles. Also he says that whatever is best treated algebraically is postponed.

The schoolmaster must have welcomed his rich supply of exercises, not only in each chapter, but in the sets of miscellaneous exercises and papers of the Oxford Local, Cambridge Local, and the Society of Arts Examinations. The answers to all are grouped together at the end of the book. The scholars, on the other hand, must have been bored by the long calculations he set. There is a multiplicand of 16 digits with a multiplier of 14, and a dividend of 24 with a divisor of 9 digits.

The topic most stressed is decimals. He urges their wider use. He not only explains them without

recourse to vulgar fractions, but puts them before the vulgar fractions in the book. There is a chapter devoted to the metric system of weights and measures, and among his tables is the proposed decimal coinage for England.

£1	= 10 florins	fl.
1 florin	= 10 (name not fixed, say chequers).	
1 (?)	= 10 (new) farthings.	

CHAPTER IX

THE EXERCISE BOOK

IF the sixteenth century was the formative period in popular arithmetic, the nineteenth was surely the century of criticism. We saw in the last chapter the method of teaching by rote and the errors arising from it condemned, and attention drawn to improvements which were overdue. The teaching of principles, we were told, should take the place of directions for special cases, more exercises for practice and better facilities for correcting for the teacher were wanted, and in the case of a particular rule, the statements in the Rule of Three and all its derivatives—a great part of the subject—should convey a meaning.

The following criticism of teaching, however, is from a different angle. It is by William Wallis, a teacher at Bridgwater in Somerset, and appears in a book entitled *An Essay on Arithmetic*.[1] The book is undated, but perhaps it was published before 1800.

> And I have seen a *Fair-Book* (as 'tis called) of a young man's, about 17 years of age, who had been 6 years at School, but never went through that rule (of three). In the same book I found 132 questions in Reduction, in the working of them were 2680 figures, which might have been better done in 500, so that there were 2180 superfluous ones. In another rule I saw an example, in which were 174 figures, but which might have been done in 23; and one of 80 that might have been done in 12: In general I have found in the boys' books 3 or 4 times as many figures as need be. These methods have so far hindered their advances in learning, that amongst 30 scholars, since I came hither, I have

[1] See *Arithmetical Books*, De Morgan, 1847.

> not found one that understood a rule beyond division, though some of them were 14 or 15 years of age, and had been kept at school ever since they were capable of being taught.

In an age of such disproportionate opinion of mechanical accuracy that it could tolerate the setting of an example with fourteen figures in the multiplier, or a problem the answer to which was expressed in thirty-five figures, a little wastage more or less in figuring would have little significance in the eyes of some. Also it sometimes happens even now that a teacher who has little knowledge upon which to draw has disciplinary sense enough to fill a pupil's time. Whatever the reason, in this case we find a teacher who knew of better things.

The passage quoted introduces the fairly common practice throughout the nineteenth century of using the exercise book, not as it appears to have been used in this instance, but as the pupil's manuscript text. Even in the late years of the century, when schools which could afford them were well supplied with books, the practice held. For the art of writing arithmetic for school children lags behind the teaching of it by word of mouth. Then as now the lesson gave knowledge and inspiration, and the text-book the supply of exercises for practice. Some will remember the practice in the eighties of last century soon after girls' schools were opened, and women were trying to give girls an education similar to that given to their brothers. Pupils were given an exercise book of twenty-four pages of good quality paper in addition to the larger class book. The lesson proceeded on the lines laid down by De Morgan, that is, an example was used to introduce the topic with argument and explanation, then the principle involved in it was driven

home with two or three examples and lastly the type problem was copied faithfully from the blackboard into the good exercise book. The type problem replaced the definition, rule, and illustration of yore. Nothing but type problems or an occasional note was entered in this book, and it became the pupil's texts term by term. The work was neat, well arranged, and written in black ink on ruled lines. The teaching was leisurely and perhaps did not go far. Stocks was often the goal of girls whose mothers in the seminaries for young ladies in the sixties had learnt the first four rules, simple and compound, and the aliquot parts for practice. What was in the syllabus, however, was taught carefully, the principles were made plain, and the written work was in a style which came of complete and well-expressed statements. It was a foundation upon which one could build, and it helped many of the pupils to fuller achievement.

A generation earlier in the grammar schools, academies, and public schools a similar book was used. It contained more pages and was better bound, the writing was more copperplate, the ruling in red ink, and the first page was adorned with scrolls and patterns around the name, subject, and date. Often the master embellished the first page. The work went farther, as befitted boy's work.

An excellent account of exercise books earlier still is in an article entitled *Grammar School Arithmetic a Century Ago* by R. S. Williamson, M.A., published in May 1928 in *The Mathematical Gazette.* It describes some mathematical exercise books compiled by a pupil at a village grammar school in Yorkshire during the years 1817 to 1822. The books were larger than those just described, having from sixty-eight to one hundred and eighty-two pages, and were made to last

a year apiece. The larger books were bound in stiff backings with leather over the hinges and corners, and in all of them the pupil had to draw fine pencil lines for his writing and figuring. The work is described as having large florid copperplate headings, exceedingly legible small-hand writing, and artistically formed figures and an abundance of red ink lining. It was evidently the pupil's text and was draughted on the classic lines of definition, rule, and illustration. De Morgan's recommendations for teaching made ten years later may appear to some to be much ado about nothing, but he spoke for the pupil when he deposed definition and rule. Here is a definition written laboriously by a boy of eleven years of age in 1817.

> Inverse proportion is when more requires less, and less requires more. More requires less is when the third term is greater than the first, and requires the fourth term to be less than the second. And less requires more is when the third term is less than the first, and requires the fourth term to be greater than the second.

In other details also the work is performed in the ways described in the books to date. There are the superabundance of figuring, the overstressed accuracy —*e.g.*, the earth's surface is computed to the tenth decimal place of a square mile—the duodecimal area, the problem set in verse, the misstatement in proportion, the decimal comma, and useful mensuration.

The books are not complete, but as far as they show, the work in each year is this:

Age 11. Inverse proportion, double Rule of Three, practice, simple interest, commission, purchasing stocks, brokerage, more simple interest, compound interest, rebate or discount, equation of payments, barter, profit and loss.

Age 12. Conversion of English units of measure into Italian, Portuguese, Dutch, German, Irish.

Arithmetical and geometrical progressions, permutations, vulgar fractions, including single Rule of Three direct and inverse, decimals.

Age 13. Mensuration.

Age 14. Arithmetic of the building and decorating trade, involving use of all the units of bricklayers, masons, carpenters, slaters, plasterers, painters, glaziers, paviours, and plumbers, with further problems on vaulted and arched roofs, spherical dome, vacuity by an elliptical groin, specific gravity. Distance determined by the velocity of sound.

Age 15. Geometrical problems.

The evidence of the fair-book and its successor, the good exercise book, points to the fact that during the nineteenth century more store was set in schools on the personal teaching of the subject than on teaching by the printed book. Who, looking at the books of the time and admitting them clear to the teacher, could imagine them other than distasteful to the pupil? At the end of the century, when the technical faults were eliminated, they became precise to a fault, shorn of all human appeal, and almost unreadable by a child. As the subject survived, it would seem that it was due to the oral teaching of it, for which in some measure the Training Colleges, by this time teaching what matter to choose and the method of handling it with a class, can take the credit. As the century wore on men and, later, women from the Training Colleges went to teach in primary and endowed schools, bringing a happier handling of the subject even if a tendency to more standardized teaching. Yet the nineteenth century ended with an opinion fairly generally expressed that arithmetic was the least liked of all school subjects.

CHAPTER X

ARITHMETIC IN RECENT YEARS

In the years considered in the previous chapters the most valuable and far-reaching additions to arithmetic were the decimal fractions and logarithms. Decimal fractions were accepted at once, and featured in the popular books. The progress of arithmetic in other directions was unhurried. Recent years, however, have seen a greater interest taken in education. The many opportunities for discussion given by teachers' associations, the conferences between teachers and examiners, and the provision of lectures for teachers, have all helped to disseminate ideas and bring about a speedy adoption of the developments presently enumerated in this chapter. Thus into the last fifty years have been crowded modifications of the processes of calculation, an increased range of application of the arithmetical rules, different methods of displaying data and results, and alterations in the class teaching of the subject. Two other new movements, namely, the custom of sending children to school at a much earlier age than before and the better equipment of school buildings, are in a great measure responsible for the important advance of modern years, the approach to the subject through practical work.

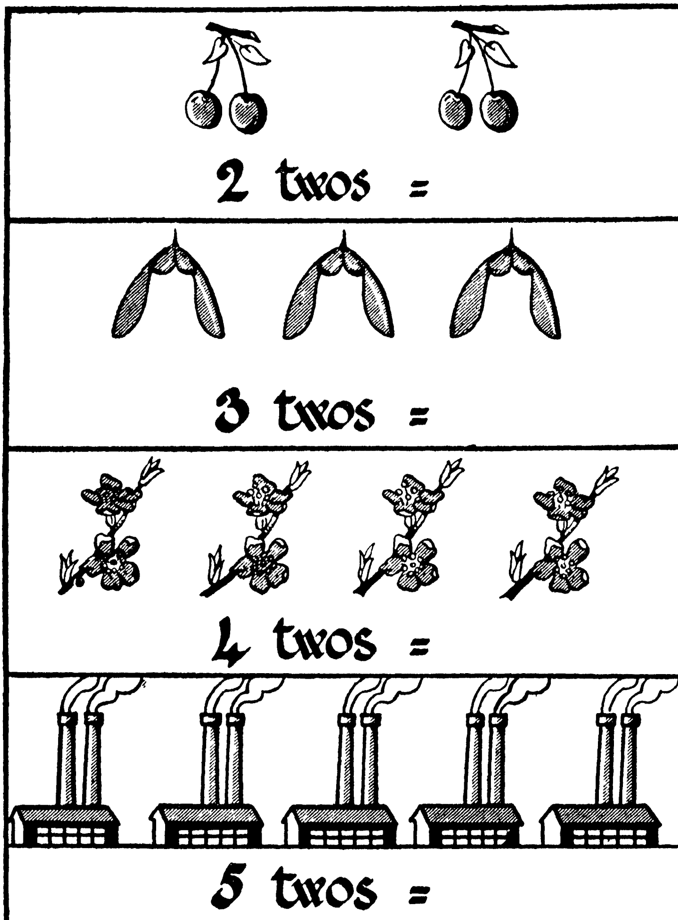
The younger children coming into the schools were little more than infants. It was evident that formal arithmetic was unsuitable for them. In arranging the course in number work it was assumed that the recognition of a unit and of several units in sequence was taught in the home. This had always been taught in

infancy; and the toe games for it, which are in reality no more than preliminary number work, had come down through the centuries. The work in school began with the grouping of similar units. Counting, adding, taking away, and multiplying were all taught with real things. For the child who can use a pencil and read simple words there are now books for such work with sets of pictures of familiar objects. In this way, happily and as befits their years, children are led up to the more formal arithmetic. Foreign schools showed us these methods in practice, but in the books of the nineteenth century we have found that both Lardner and Sang advocated the use of common objects for the easy learning of the early rules. Lardner, in particular, made very full use of counters in his explanations of the different processes of calculation.

A few years later many school buildings were provided with science laboratories, giving the opportunity of doing preliminary practical work in mensuration also. The laboratories were furnished with cubes which could be handled for learning dimensions, areas, and volumes of rectangular solids, and with gas-jars, burettes, etc., to enable the volumes of the non-rectangular solids and the capacities of vessels to be found. In the geometry room also practical work found a place, and the early work in mensuration included finding the dimensions required for computation by direct measurement.

For the second time in its history arithmetic enlarged its borders, this time to include practical work as an introduction to certain elementary rules. The scope of the subject was now so widened that we do not find the whole school course of pure, applied, and practical arithmetic between the covers of one book.

COUNT



Introduction to Multiplication

From *The Child's First Number Book*, by P. B. Ballard, M.A., D.Lit.

By courtesy of the University of London Press

There are many small changes to notice. To-day much less work is done on weights and measures, and much less time is spent on lengthy calculations for the sake of accuracy and on involved vulgar fractions. General rules and principles are taught instead of a multitude of special cases. The teaching on money matters has become more varied. The financial page of the newspaper is explained, and made the basis of work in stocks and shares and in exchange between countries. The budget and rates and tax papers are drawn upon for exercises, and the work bench brings its problems. Such collected information as trade and traffic returns and series of readings from the news or from laboratory experiments give material for graphs and statistics. Logarithms have come to be used in calculations.

In one rule we find a strange reversion to old time practice. It took over a hundred years to remove the old method of division from the books. The massed figures and the crossing out of the used figures were both serious objections to it; and two writers, at least, urged the use of the more plainly stated new (long) division. Rather more than a century after the old division had finally died out, about the middle of the nineteenth century, the custom began of performing the reduction of vulgar fractions in a mass of crossed-out figures. The method is known to all to-day. Contrary to popular belief, this cancelling of numbers in fractions does not appear to be a survival of an old method. In the past multiplication, division, and extraction of roots showed the crossing out of figures, but the problems in vulgar fractions were very elementary, and the fraction was written afresh at each stage of the simplification.

Many readers will remember that the teaching of

multiplication used to begin with the learning of tables in the form:

3	1's are	3,
3	2's are	6,
3	3's are	9,
3	4's are	12, etc.,

where the number 3 was operating on and trebling each of the numbers 1, 2, 3, . . . 12 in turn.

The practice still holds in many schools. There is, however, a tendency to-day to reverse the factors in each line and teach a table of the multiples of three. In schools where practical work precedes the committal of tables to memory the latter form is generally approved. The child lays down 3 sticks, another 3 sticks, and another 3 sticks, and so on, counting at each step, and so builds up a table of multiples of 3. To lead up to the other form of table he would have to lay down 1 stick three times, then, beginning afresh, place three groups of 2 sticks, and, again beginning afresh, three groups of 3 sticks, and so on, and this work, to a young child, would appear to be a number of unconnected operations.

For a time unitary method replaced the method of proportion for the solution of problems.

In place of the old Rule of False Position there is now the shorter and more certain solution of problems by the use of x. After being in abeyance for a time the methods of checking results are again taught. The preliminary survey of the problem to form an approximate idea of the answer to it appears to be a modern recommendation.

Our attitude towards the long numbers has changed, and an approximation, properly controlled according to the purpose it is to serve, is preferred to the very

high numbers correct to the unit or the decimal fraction carried to many places.

The teaching of decimals has never been settled. During three hundred years there have been eight different methods of writing decimals, two or three methods of multiplication, and at least six methods of division. Almost without exception in division the directions have been to change the problem set into one with different values of dividend and divisor. The most commonly used preparations for division are the following:

$$(1)\quad \frac{23\cdot876}{\cdot19} = \frac{23876 \div 10^3}{19 \div 10^2} = \frac{23876}{19} \div 10,$$

$$(2)\quad \frac{23\cdot876}{\cdot129} = \frac{2\cdot3876 \times 10}{1\cdot29 \div 10} = \frac{2\cdot3876}{1\cdot29} \times 10^2,$$

and the standard form of to-day. In every case, except the standard form, further directions are needed for replacing the decimal point in the quotient; and in every case directions are needed for giving a true value to the remainder. Knowing that each problem so approached entails two or more pieces of work, custom has always been to avoid using the decimal point where it is given.

Our arithmetical text-books have always shown the decimal point hedged around with difficulties. The difficulties are still with us. They are often discussed, but the remedies suggested have not as yet been tested on any great scale.

The criticism of the decimal point, as usually placed, has come generally from workers in logarithms. De Morgan suggested that the decimal dot should be considered "as part and parcel of the units place." In his illustration he did not alter the position of the dot, but asked his readers to interpret it differently.

More recently it has been suggested that we remove the decimal point from the right-hand side of the unit figure, and put instead a circumflex accent over the unit figure. The benefit claimed is the simplification of the rule of the characteristic; which would become, in De Morgan's wording, "The characteristic of (the logarithm of) a number is the number of places by which the first significant figure is distant from the unit's place; and is positive when that first figure falls to the left, negative when to the right." Thus, using the circumflex accent to mark the unit, the characteristic of the logarithm of $12\hat{3}456$ would be 3, and the characteristic of the logarithm of $\hat{0}00029$ would be $\bar{4}$.

The decimal point was not the only mark of the decimal, nor even the original mark. The first seven of the following methods of writing decimals have been in use at different times.

(A)	$123'4''5'''$	(F)	12,345
(B)	$12\vert\underline{345}$	(G)	12·345
(C)	12|345	(H)	$12\overset{\vert}{\underset{\vert}{3}}45$
(D)	$12\underline{345}$		
(E)	$12\dot{3}4\dot{5}$	(J)	$12\hat{3}45$

Some of these illustrations, namely, A, D, E, H, J, show an uninterrupted line of figures. Others, B, C, F, G, show the series broken in two. B shows the symbol used for dividing by ten in the days before decimals were known, when its use was to separate the quotient from the remainder. It carried over into decimals the idea of separation, this time of integers from fractions. Its derivative in C, and the comma and dot in F and G respectively, perpetuate this idea of separation, and

lead us to look for differences rather than likenesses in the two parts of the series.

Stevinus, in the form shown in A, and Jager, who used the form in E, laid more emphasis on the series being the extension of the Hindu-Arabic notation. Cocker gave an illustration, H, showing the place value of the figures in the series, but beyond that he did not develop the idea in any way or even refer to it again in either of his books. The modern suggestion, J, is, in essence, Cocker's.

1 2 $\overset{\wedge}{3}$ 4 5

Hundreds Tens Tenths Hundredths

This notation gives an unbroken series of figures. It brings a certain symmetry into the series which the decimal point, as usually placed, masks; and it emphasizes the unit figure which, alone of all the figures in the series, has simply its own intrinsic value.

Another of the changes in recent years concerns questions and their answers. In all the early books the answer was put at the end of the question. These books, however, were for older or private pupils. With increased class teaching came the question of the advisability of allowing the pupil to see the answers. One of the earliest writers to separate answer from question was J. Carver, who did so in *The Master's and Pupil's Assistant*, 1815. De Morgan, who, however, did not himself adopt the plan said of him: "The author of this work, dependent as the sale of it was on teachers, has had the sense and courage to say that questions with answers are for the benefit of the masters and the injury of the pupils." As the century wore on other teachers came to hold the same opinion,

and gradually the answers to problems were eliminated from the pupils' books. The pupils' book was no longer a text-book, but consisted of sets of exercises without answers. At the present the tendency is to restore the answers to questions, and also the text of arithmetic to books for pupils.

Through the four centuries we have considered the standard arithmetical text-books have been written in dialogue by Recorde and Hylles, in verse by Hylles, Vyse, and Chappell, with full and well illustrated explanations by Wingate, Ward, and Bonnycastle, and with the concise notes of the handbooks for teachers. Of minor ventures there are the alphabetically arranged rules of Lowe, and the 'epitome.' Arithmetic has even been written in question and answer in the years when that method of teaching was in vogue. Except for the occasional anecdotes of Ward, the story-book has not yet been written. But, whatever the mode of presentation, the reader of arithmetic always finds himself in agreement with Recorde's scholar, who said, "This is marvellous, methinks, that such great matters may so easily be achieved by this Art, which heretofore I ever thought had been impossible."

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